

The University of Chicago

THE MINIMUM OF A DEFINITE
INTEGRAL WITH RESPECT TO
UNILATERAL VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1934

By

JULIAN DOSSY MANCILL

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
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INTRODUCTION

The problem of unilateral (one-sided) variations in the calculus of variations to be considered here is that of finding in a prescribed class of arcs with equations of the form

$$x = x(t), \quad y = y(t) \quad (t_1 \leq t \leq t_2),$$

which join the two fixed points (x_1, y_1) , (x_2, y_2) and lie entirely on one side of a fixed boundary curve, one which minimizes the integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y') dt.$$

The problem for three dimensions is the same except that in this case the class of arcs lies entirely on one side of a prescribed surface. Probably the first illustration of the problem of unilateral variations was discovered by Goldschmidt¹ in 1831, in which the curve generating a surface of revolution of minimum area is composed of a piece of the axis of revolution together with two line segments perpendicular to it. In this example the boundary is the axis of revolution.

In 1871 Todhunter² gave the inequality $T \leq 0$ (≥ 0) which replaces the Euler equation $T = 0$ for a piece of the minimizing

¹Determinatio superficiei minimae rotationis curvae data duo puncta jungentis circa datum axem ortae, Gottingen Prize Essay (1831).

²Researches in the calculus of variations, London, 1871, p. 13.
(89)

curve lying along the boundary of the region of admissible curves. Also Weierstrass obtained the condition $\mathfrak{G}(x, y, x', y', \zeta', \eta') = 0$, where (x', y') gives the direction of the extremal and (ζ', η') gives the direction of the boundary at the points where the minimizing curve meets the boundary. If the problem is regular on the boundary, this condition implies tangency and no corners are present. For this case Bliss¹ has derived sufficient conditions for a minimum. The regular problem of unilateral variations in three dimensions has been treated by Bliss and Underhill.²

The following pages are devoted to a study of these problems without the hypothesis of regularity. Part I is devoted to certain frequently used preliminary definitions and theorems.

Part II is devoted to a rather exhaustive treatment of the plane case by comparatively general methods. The analogue of the Jacobi condition is derived for each arc of the minimizing curve along which $T = 0$, under two sets of hypotheses relative to the behavior of the $\mathfrak{G}$ -function in a neighborhood of the point where the minimizing curve leaves the boundary. There is no restriction on the length of the arcs of the boundary in common with the minimizing curve along which $T \neq 0$ as the sufficient conditions of Bliss show. It is shown that the usual corner conditions are necessary at the corners of the boundary curve at which the arc beyond the corner is directed towards the left of the one preceding the corner in the case where the region of admissible curves lies to the left of the boundary. These conditions are not necessary

¹Sufficient conditions for a minimum with respect to one-sided variations, Transactions of the American Mathematical Society, V (1904), pp. 479-92.

²The minimum of a definite integral for unilateral variations in space, Transactions of the American Mathematical Society, XV (1914), pp. 291-310.

at corners of the boundary at which the arc beyond the corner is directed to the right of the one preceding the corner. In the former case the function Ω_0 is different from zero at the corners of the boundary (including the points where the minimizing curve meets the boundary), provided at least one of the two arcs meeting there has $T \neq 0$ along it. Also a set of sufficient conditions for a strong proper relative minimum is derived which allows the Weierstrass $\mathcal{G}$ -function to vanish along a finite number of arcs or at a finite number of points of the boundary curve for certain unique directions. The fact that this set of conditions is sufficient for a proper relative minimum follows from the statement of a more general sufficiency theorem. Finally, it is observed that the whole treatment can be extended to the case when the minimizing curve has a finite number of distinct segments in common with the boundary.

Part III has to do with a treatment of unilateral variations in three dimensions, following the same general outline as in Part II except that the treatment of this case is not so exhaustive. The analogue of the Jacobi condition is derived for the extremal arcs of the minimizing curve interior to the admissible region and for the surface extremal arc on the boundary surface in the case where the surface extremal is not a space extremal. This condition, however, is proved only under rather restrictive hypotheses in neighborhoods of the two points where the minimizing curve meets the boundary surface. It is also shown that at these two points the usual corner conditions are satisfied by the minimizing curve and that the function Ω_0 is not zero provided the space extremal arcs are not tangent to the surface at these two points and the surface extremal is not a space extremal. A general sufficiency theorem is proved which is analogous to the

one in the plane case. Then a set of sufficient conditions is given for a strong proper relative minimum. However, these conditions contain the same restrictive assumptions necessary for the proof of the necessary conditions.

PART I

PRELIMINARY DEFINITIONS AND THEOREMS

We shall give here certain definitions and theorems which will be of frequent use in the following pages. In Parts I and III we shall understand that the variables displayed in a function are n -partite (where, in Part III, $n = 3$). Also we shall make use of the well-known summation convention of tensor analysis whenever the complexity of the notation seems to require it. The matrix notation, omitting the subscripts, will be used if there is no danger of confusing this with the tensor notation.

The integral to be considered has the form

$$I = \int_{t_1}^{t_2} F(x, x') dt.$$

The function $F(x, x')$ is assumed to have the following properties in a region R' , consisting of all points (x, x') such that x is in a region R and not all the components of x' are zero:

- (A) $F(x, x')$ is of class C^{IV} ,
- (B) $F(x, hx') = hF(x, x')$ ($h > 0$).

THEOREM 1:1. The function

$$\mathcal{E}(s, p) \equiv \mathcal{E}(x, x', p) \equiv F(x, p) - p_1 F_{x_1}(x, x'),$$

taken along an extremal

$$E_0: x = x(s),$$

where s is the length of arc along E_0 , has the derivative

$$\mathcal{E}_s(s, p) \equiv \Omega_s(x(s), x'(s), p) \equiv x_1' F_{x_1}(x, p) - p_1 F_{x_1}(x, x').$$

This follows immediately from differentiating $\mathcal{G}(s, p)$ with respect to s and making use of the extremal property of E_0 .

We shall say that a curve E is positively strong at an element (x, x') if $F_1(x, x') \neq 0$ and $\mathcal{G}(x, x', p) > 0$ for all directions $(p) \neq (hx')$ ($h > 0$). A curve E is said to be an extremaloid if it satisfies the Euler equations $(d/dt)F_{x'} = F_x$ between corners and satisfies the corner conditions,

$$(1) \quad F_{x'}(x, \bar{x}') - F_{x'}(x, x') = 0,$$

at each corner, where (x') and $(\bar{x}')$ determine the two directions of E at the corners. An extremaloid is said to be positively strong if each extremal arc composing it is positively strong except at the end points, and the function F_1 remains different from zero on both sides of the corners on the extremaloid.

The next theorem and the two lemmas leading up to the theorem are stated here as proved by Graves,¹ except for a slight modification of notation.

LEMMA 1:1. If an extremal E_0 ceases to be strong at a point x_1 , but still has the quadratic form $Q(x_1, x'_1; \eta)$ regular,² then there exists an admissible direction $(\bar{x}')$ which with the direction (x'_1) of E_0 at x_1 satisfies the corner conditions (1).

In view of this lemma, we shall say that the direction $(\bar{x}_1')$ is a continuation direction of (x_1') at the corner x_1 .

LEMMA 1:2. If an extremal E_0 has the properties:

(1) E_0 has the unique continuation direction $(\bar{x}_1')$ at the point x_1 ;

¹Discontinuous solutions in space problems of the calculus of variations, American Journal of Mathematics, LIII (1930), pp. 9-13. Hereafter we shall refer to this paper as "Discontinuous solutions."

²The quadratic form $Q(x, x'; \eta) = \eta_1 F_{x_1' x_1'} \eta_1$ is said to be regular in a set of values (x, x') if it does

(2) the quadratic form Q is positive regular at (x_1, x_1') ;

(3) $F_1(x_1, \bar{x}_1') \neq 0$;

(4) $\Omega_0(x_1, x_1; \bar{x}_1') \neq 0$;

(5) the family $x = \varphi(t, a, b)$ of extremals contains E_0

for $a_1 = a_{10}, b_1 = b_{10}$;

(6) the parameter value t_1 corresponds to the point x_1 on E_0 ;

then there are neighborhoods $N(a_0, b_0)$ and $N(t_1)$ such that for each (a, b) in $N(a_0, b_0)$ there is one and only one set of functions $[t(a, b), \bar{x}'(a, b)]$ having $t(a, b)$ in $N(t_1)$ and the direction $(\bar{x}'(a, b))$ different from $(\varphi'[t(a, b), a, b])$, and satisfying the corner equations

$$F_{x_1}[\varphi(t, a, b), \bar{x}'(a, b)] - F_{x_1}[\varphi(t, a, b), \varphi'(t, a, b)] = 0, \\ \bar{x}_1' \bar{x}_1' - 1 = 0.$$

This simply says that if an extremal E_0 has a unique continuation direction at a point x_1 , the same will be true of each member of a family containing E_0 sufficiently near E_0 for some value of t in a neighborhood of t_1 , under the conditions as stated in the lemma.

THEOREM 1:2. Let E be an extremaloid which is positively strong and which has $\Omega_0 \neq 0$ at each corner. Suppose also that the extremaloid E is uniquely determined by each of its elements (x, x') . Let $x = \varphi(t, a, b)$ be equations of the $(2n - 2)$ -parameter family of extremaloids containing E for $a_1 = a_{10}, b_1 = b_{10}$. Then there is a neighborhood of (a_0, b_0) in which every corresponding extremaloid of the family is positively strong.

not take opposite signs for different values of its arguments and vanishes only when $(\gamma) = (0)$, or when the directions (γ) and (x') are in the same line.

THEOREM 1:3. If the family of curves

$$(2) \quad x = \phi(s, \lambda)$$

satisfies

$$I^{(x_1)} \phi_{1\lambda} \equiv 0,$$

where $I^{(x_1)} \equiv F_{x_1} - (d/ds)F_{x_1}'$, then the integral

$$\int_{s_0}^{s_1} F(\phi, \phi_s) ds,$$

in which $s_0 = s_0(\nu)$, $s_1 = s_1(\nu)$, $\lambda = \lambda(\nu)$, has the derivative

$$F_{x_1}, x_{1\nu} \Big|_0^1, \text{ where } x_{1\nu} \equiv \phi_{1s}s_\nu + \phi_{1\lambda}\lambda_\nu. \quad ^1$$

THEOREM 1:4. If the family

$$(3) \quad x = \phi(t, \mu)$$

also satisfy $I^{(x_1)} \phi_{1\mu} \equiv 0$ and adjoin the family (2) for $s = s_1(\nu)$, $\lambda = \lambda(\nu)$, $t = t_1(\nu)$, $\mu = \mu(\nu)$, and if corresponding arcs satisfy the corner conditions (1) at their points of adjunction, then the function

$$I(\nu) \equiv \int_{s_0}^{s_1} F(\phi, \phi_s) ds + \int_{t_1}^{t_2} F(\phi, \phi_t) dt,$$

where $t_2 = t_2(\nu)$, has the derivative

$$I'(\nu) = F_{x_1}, x_{1\nu} \Big|_0^2,$$

$$\text{where } x_{1\nu} \Big|_0^0 \equiv \phi_{1s}s_\nu + \phi_{1\lambda}\lambda_\nu, \quad x_{1\nu} \Big|_2^2 \equiv \phi_{1t}t_\nu + \phi_{1\mu}\mu_\nu.$$

¹Bliss and Underhill, The minimum of a definite integral for unilateral variations in space, Transactions of the American Mathematical Society, XV (1914), p. 300. Hereafter we shall refer to this paper as Bliss and Underhill.

We know that the function $I(\nu)$ has the derivative

$$I'(\nu) = F_{x_1, x_1 \nu} \Big|_0^1 + F_{x_1, x_1 \nu} \Big|_1^2$$

by Theorem 1:3. But by hypothesis $\varphi[s_1(\nu), \lambda(\nu)] \equiv \phi[t_1(\nu), \mu(\nu)]$, and therefore $\varphi_{s_1 s_1 \nu} + \varphi_{\lambda \lambda \nu} \equiv \phi_{t_1 t_1 \nu} + \phi_{\mu \mu \nu}$. That is,

$$\begin{aligned} I'(\nu) &= F_{x_1, x_1 \nu} \Big|_0^2 - x_{1\nu} [F_{x_1, (\phi, \phi_t)} - F_{x_1, (\phi, \phi_s)}] \Big|_0^1 \\ &= F_{x_1, x_1 \nu} \Big|_0^2, \end{aligned}$$

since the corner conditions are satisfied at the point 1. Obviously this holds for any number of families adjoined as described in the theorem.

If the curves of the first family all pass through a fixed point 0 for $s = s_0(\nu)$, $\lambda = \lambda(\nu)$, then the expression $F_{x_1, x_1 \nu} \Big|_0^1$ vanishes at the point 0, since the derivatives of the first family vanish there. If the curves of the second family similarly pass through a fixed point 2 for $t = t_2(\nu)$, $\mu = \mu(\nu)$, then $I(\nu)$ is a constant. If in case of tangency of the two families, the second family is independent of μ , then it will be simply an enveloping curve of the first family. Also, if in this case $t_2(\nu)$ is a constant, the second integral in $I(\nu)$ will be simply the value of I taken along the enveloping curve from the point 1 to a fixed point 2, and the function $I(\nu)$ will again be a constant.

THEOREM 1:5. If at a point (x, x') in R' we have $\mathcal{E}(x, x', p) \geq 0$ for all $(p) \neq (0)$, while $\mathcal{E}(x, x', \bar{x}') = 0$, then also

$$F_{x'}(x, \bar{x}') - F_{x'}(x, x') = 0.$$

At such a point $\mathfrak{E}(x, x', p) \equiv \mathfrak{E}(p)$ takes on its minimum value zero for $(p) = (h\bar{x}')$. Thus we have

$$\mathfrak{E}_p(\bar{x}') = F_{x'}(x, \bar{x}') - F_{x'}(x, x') = 0.$$

PART II
UNILATERAL VARIATIONS IN THE PLANE

2:1. Statement of the problem. Suppose that the curve E_{03} joining the two fixed points 0 and 3, whose minimizing properties are to be studied, is composed of the three arcs

$$\begin{aligned} E_{01}: \quad x &= x_0(t), \quad y = y_0(t) & (t_0 \leq t \leq t_1), \\ E_{12}: \quad x &= \zeta(\sigma), \quad y = \eta(\sigma) & (\sigma_1 \leq \sigma \leq \sigma_2), \\ E_{23}: \quad x &= x_3(t), \quad y = y_3(t) & (t_2 \leq t \leq t_3). \end{aligned}$$

The arcs E_{01} and E_{23} are assumed to be of class C^1 while the arc E_{12} is supposed to be of class C^n on the extended interval $\sigma_1 - h \leq \sigma \leq \sigma_2 + h$, where $h > 0$ and σ is the



length of arc along B from some fixed point. We shall suppose that the curve E_{03} does not intersect itself and that the boundary B has only the points 1 and 2 in common with the curves E_{01} and E_{23} which are on the same side of B.

The function F in the integral

$$(1) \quad I \equiv \int F(x, y, x', y') dt$$

is assumed to have the following property in addition to those enumerated at the beginning of Part I:

(C) $F_1(x, y, x', y') \neq 0$ along the curve E_{03} including both sides of corners.

It is the purpose of Part II to develop necessary conditions and sufficient conditions that the curve E_{03} must satisfy in order

to minimize the given definite integral (1) in the class of admissible curves. The class of admissible curves is composed of those curves E which satisfy the following conditions:

- (1) E passes through the fixed points 0 and 3;
- (2) E lies on the same side of the boundary as the arcs E_{01} and E_{23} ;
- (3) E is of class D^1 ;
- (4) each element (x, y, x', y') of E is interior to the region R^1 .

The notation will be constructed naturally as the theory requires it.

2:2. Review of necessary conditions already known. We suppose, as usual, that the curve E_{03} minimizes the integral I and wish to formulate its properties.

I. If E_{03} minimizes I , then the arcs E_{01} and E_{23} satisfy the equations $(d/dt)F_{x'} = F_x$, $(d/dt)F_{y'} = F_y$.¹

Since we assumed that the function F_1 is not zero along the arcs E_{01} and E_{23} , it follows from the well-known differentiability conditions that these arcs are of class C^n . In this case we may say that E_{01} and E_{23} satisfy the Weierstrass form of the Euler differential equations

$$T \equiv F_{xy'} - F_{x'y} + F_1(x'y'' - x''y') = 0,$$

where

$$F_x - \frac{d}{dt}F_{x'} = y'T, \quad F_y - \frac{d}{dt}F_{y'} = -x'T.$$

II. If E_{03} minimizes I , then along the boundary we have $T \leq 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$.

¹Bolza, Lectures on the calculus of variations, University of Chicago, 1904, p. 122. Hereafter this book will be referred to as "Lectures."

In the proof of this condition it is assumed that as we go along the boundary from the point 1 to the point 2, i. e., in the positive direction of the minimizing curve, the region of admissible curves lies to our left. In the contrary case the direction of the inequality sign should be reversed, i.e., we would have $T \geq 0$ along the arc B_{12} .¹ This condition states that the arc B_{12} of the boundary may or may not be an extremal arc, or that B_{12} may contain arcs of extremals and arcs which are not extremals. We shall find it convenient to consider, in order, the two cases; (1) $T < 0$ along B_{12} ; (2) $T = 0$ along B_{12} . Then we shall be able to consider certain combinations of these two cases under modified hypotheses concerning the boundary curve. We shall assume then, until the contrary is explicitly stated, that $T < 0$ along B_{12} .

III. If E_{03} minimizes I , then at the points 1 and 2 where the extremal arcs E_{01} and E_{23} meet the boundary respectively, we have $\mathcal{G}(x, y, x', y', \zeta', \eta') = 0$.²

In the case of the regular³ problem this condition says that the arcs E_{01} and E_{23} are tangent to the boundary at the points 1 and 2 respectively. This is the case considered by Bliss already referred to in the introduction. We shall see that his treatment follows naturally as a special case of the problem here treated.

IV. If E_{03} minimizes I , then $\mathcal{G}(x, y, x', y', p, q) \geq 0$ for each element (x, y, x', y') along E_{03} and all $(p, q) \neq (0, 0)$.⁴

¹Bolza, Lectures, p. 148.

²Bolza, Vorlesungen über Variationsrechnung, Berlin, 1909, p. 396. Hereafter this will be referred to as "Vorlesungen."

³A problem in the calculus of variations is said to be regular in case $F_1(x, y, p, q) \neq 0$ for all (x, y) in R and all $(p, q) \neq (0, 0)$. In this type of problem no corners occur.

⁴Graves, A proof of the Weierstrass condition in the calculus of variations, to be published in the American Mathematical

2:3. Families of extremaloids associated with E_{03} . We shall show in this section the existence of certain families of extremals which may, under appropriate hypotheses, be pieced together to form families of admissible extremaloids tangent to the boundary.

THEOREM 2:1. The conditions

$$(2) \quad \begin{aligned} F_x(x, y, x', y') - F_x(x, y, \zeta', \eta') &= 0, \\ F_y(x, y, x', y') - F_y(x, y, \zeta', \eta') &= 0 \end{aligned}$$

are satisfied at the points 1 and 2 along E_{03} .

This follows as a corollary to Theorem 1:5 on account of the conditions III and IV.

THEOREM 2:2. If $T < 0$ along B_{12} and $\mathcal{E}(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$ and all $(p, q) \neq (0, 0)$, then the function

$$\begin{aligned} \Omega_0(\zeta, \eta, \zeta', \eta', x', y') &\equiv \zeta' F_x(\zeta, \eta, x', y') + \eta' F_y(\zeta, \eta, x', y') \\ &\quad - x' F_x(\zeta, \eta, \zeta', \eta') - y' F_y(\zeta, \eta, \zeta', \eta') \end{aligned}$$

is positive at the point 1 and negative at the point 2, if (x', y') denotes the direction of E_{01} at the point 1, and of E_{23} at the point 2 in the respective cases.

Since the function $\mathcal{E}(\zeta, \eta, \zeta', \eta', x', y')$ vanishes at the point 1 for the direction (x', y') along E_{01} , we have

$$\begin{aligned} \mathcal{E}_0(\zeta, \eta, \zeta', \eta', x', y') \Big|_{\sigma=\sigma_1} &= \Omega_0(\zeta, \eta, \zeta', \eta', x', y') + T(x'\eta' - y'\zeta') \\ &= \Omega_0 + T \sin(\theta - \phi) \\ &\geq 0, \end{aligned}$$

Monthly. I was able to make a somewhat complicated field proof of this condition along B_{12} some time before Graves developed his very elegant and simple proof of this condition along a minimizing arc independently of the extremal property, which holds for any number of dimensions.

where $(\zeta', \eta') = (\cos \theta, \sin \theta)$ and $(x', y') = (\cos \phi, \sin \phi)$. But since (x', y') is directed to the right of B_{12} at the point 1, we have $\sin(\theta - \phi) > 0$ and T is assumed to be less than zero along B_{12} . Therefore, Ω_0 is greater than zero at 1. The proof at the point 2 is the same except the inequalities are reversed.

In view of our hypotheses on the function F , we have the existence of a family of extremals

$$(3) \quad x = \varphi_0(t, a), \quad y = \psi_0(t, a),$$

passing through the point 0 for $t = t_0$ and containing the arc E_{01} for $a = a_0$, $t_0 \leq t \leq t_1$. The functions $\varphi_0, \varphi'_0, \varphi''_0, \psi_0, \psi'_0, \psi''_0$ are of class C^1 on $t_0 - \delta \leq t \leq t_1 + \delta$, $|a - a_0| \leq \delta$, for δ sufficiently small.¹ Likewise, there exists a family of extremals

$$(4) \quad x = \varphi_2(t, \sigma), \quad y = \psi_2(t, \sigma),$$

tangent to the boundary B at points $[\varphi_2(t(\sigma), \sigma), \psi_2(t(\sigma), \sigma)]$ for $\sigma_1 - h \leq \sigma \leq \sigma_2 + h$ and lying on the same side of the boundary as E_{01} and E_{23} . The functions $\varphi_2, \varphi'_2, \varphi''_2, \psi_2, \psi'_2, \psi''_2$ are of class C^1 for $t(\sigma) - \rho \leq t \leq t(\sigma) + \rho$ if ρ is sufficiently small. The existence of this family and the side of B on which its members lie depend upon the assumption that T is less than zero along the arc B_{12} of the boundary.

Since the corner conditions (2) are satisfied at the point 1 on E_{03} , they determine functions $\tau_1(a), p_1(a), q_1(a)$ of class C^1 such that for each point of the curve

$$C_1: x = \varphi_0[\tau_1(a), a] \equiv \tilde{x}_1(a), \quad y = \psi_0[\tau_1(a), a] \equiv \tilde{y}_1(a), \quad |a - a_0| < \epsilon,$$

and the corresponding member of the family (3) through this point

¹Bolza, Vorlesungen, p. 401.

there exists a unique admissible continuation direction $[p_1(a), q_1(a)]$ in a neighborhood of the direction of B_{12} at 1.¹ For $a = a_0$ the continuation direction coincides with the direction of B at the point 1 and hence coincides with the direction of the member of the family (4) tangent to B at 1, which we shall denote by E_1 . Also, the corner conditions (2) determine functions $\tau_2(\sigma)$, $p_2(\sigma)$, $q_2(\sigma)$ of class C^1 in a neighborhood of σ_2 , such that for each point of the curve

$$C_2: x = \Phi_2[\tau_2(\sigma), \sigma] \equiv \tilde{x}_2(\sigma), \quad y = \Psi_2[\tau_2(\sigma), \sigma] \equiv \tilde{y}_2(\sigma), \quad |\sigma - \sigma_2| < \epsilon,$$

and the corresponding extremal arc of the family (4), there exists a unique continuation direction $[p_2(\sigma), q_2(\sigma)]$ in a neighborhood of the direction of E_{23} at the point 2. For $\sigma = \sigma_2$ the direction $[p_2(\sigma_2), q_2(\sigma_2)]$ coincides with that of E_{23} at 2.

Now, from an application of the embedding theorems to the arc E_1 , we have the existence of a family of extremals

$$(5) \quad x = \Phi_1(t, a), \quad y = \Psi_1(t, a),$$

containing E_1 for $a = a_0$, $t_1 \leq t \leq t_1 + \rho$, such that the direction of each member at its point of intersection with the corner-curve C_1 is the continuation direction, $[p_1(a), q_1(a)]$, of the corresponding member of the family (3). The functions Φ_1 , Φ_1' , Φ_1'' , Ψ_1 , Ψ_1' , Ψ_1'' are of class C^1 at least for $t_1 - \rho \leq t \leq t_1 + \rho$, $|a - a_0| \leq \epsilon$, if ρ and ϵ are sufficiently small. Also, there exists a family of extremals

$$(6) \quad x = \Phi_3(t, \sigma), \quad y = \Psi_3(t, \sigma),$$

containing the arc E_{23} for $\sigma = \sigma_2$, $t_2 \leq t \leq t_3$, such that the

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 6.

direction of each member at its point of intersection with the corner-curve C_2 is the continuation direction, $[p_2(\sigma), q_2(\sigma)]$, of the corresponding member of the family (4) tangent to B. The functions $\varphi_3, \varphi_3', \varphi_3'', \psi_3, \psi_3', \psi_3''$ are of class C^1 at least for $t_2 - \delta \leq t \leq t_3 + \delta$, $|\sigma - \sigma_2| \leq \epsilon$, if δ and ϵ are sufficiently small.

It is to be noted here that the corner-curve C_2 is tangent to the boundary at the point 2, since 2 is a focal point of the family of extremals tangent to B, as follows immediately from the form of the equations of C_2 . The behavior of the corner-curve C_1 in a neighborhood of the point 1 will be discussed later.

The next two theorems relate to the condition which must be satisfied by the boundary curve B in order that the point of intersection with the corner curve C_2 (corresponding to the parameter value $\tau_2(\sigma)$), shall follow the point of tangency (corresponding to the parameter value $t(\sigma)$) on the extremals tangent to the boundary near the point 2.

THEOREM 2:3. If $\mathcal{E}_1(\zeta_2, \eta_2, \zeta_2', \eta_2', p, q) > 0$ except for (p, q) in the direction, $(\bar{x}_2', \bar{y}_2')$, of E_{23} at 2, a necessary and sufficient condition that $\tau_2(\sigma) > t(\sigma)$ in a deleted neighborhood of σ_2 is that $\mathcal{E}_1(\zeta, \eta, \zeta', \eta', p, q) > 0$ for σ in a deleted neighborhood of σ_2 and all $(p, q) \neq (0, 0)$.¹

The proof of this theorem depends upon

LEMMA 2:1. If B has

$$\begin{aligned} \mathcal{E}_1(\zeta, \eta, \zeta', \eta', p, q) &= 0, \quad (p, q) = (h\bar{x}', h\bar{y}'), \\ &> 0, \quad (p, q) \neq (h\bar{x}', h\bar{y}') \quad (h > 0) \end{aligned}$$

at a point at which $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{x}', \bar{y}') < 0$ ($\Omega_0 > 0$), then the extremal tangent to B ceases (commences) to be positively

¹For the definition of the function $\mathcal{E}_1$, see Bolza, Vorlesungen, p. 265.

strong at that point.

Consider the extremal

$$E_2: x = \varphi_2(s, \sigma_2), \quad y = \psi_2(s, \sigma_2),$$

tangent to B at the point 2, where s is the length of arc along E_2 from some fixed point. Then let

$$\mathfrak{G}(s, \alpha) \equiv \mathfrak{G}(x, y, x', y', \cos \alpha, \sin \alpha),$$

where (x, y, x', y') is the element along E_2 and $(\cos \alpha, \sin \alpha) \equiv (p, q)$. But

$$\mathfrak{G}_3(s_2, \bar{\alpha}) = \Omega_0 < 0,$$

where $(\cos \bar{\alpha}, \sin \bar{\alpha})$ is in the direction of E_{23} at 2, by Theorems 1:1 and 2:2. Thus the function $\mathfrak{G}(s, \alpha)$ is decreasing in s at s_2 for α in a neighborhood $N_{\epsilon_1}(\bar{\alpha})$. However, $\mathfrak{G}(s_2, \bar{\alpha}) = 0$ on account of the corner conditions. Therefore, $\mathfrak{G}(s, \alpha) > 0$ for s less than s_2 and in a neighborhood of s_2 and all α in $N_{\epsilon_1}(\bar{\alpha})$. Also, for every $\epsilon > 0$ and α such that α is not in the neighborhood $N_\epsilon(\bar{\alpha})$ it is true that $\mathfrak{G}_1(s_2, \alpha) > 0$. Hence, for every $\epsilon > 0$ there exists a neighborhood $N_{\epsilon_2}(s_2)$ such that for every s in $N_{\epsilon_2}(s_2)$ and α not in $N_\epsilon(\bar{\alpha})$ it is true that $\mathfrak{G}_1(s, \alpha) > 0$, since the function $\mathfrak{G}_1$ is continuous. If we choose $\epsilon \leq \epsilon_1$, then there exists a neighborhood $N_{\epsilon_2}(s_2)$ such that for $s < s_2$ and in $N_{\epsilon_2}(s_2)$ and all values of α it is true that $\mathfrak{G}_1(s, \alpha) > 0$, since the sets of values of α in N_{ϵ_1} and not in N_ϵ overlap. This proves the first part of the lemma. The second part is proved the same way except the function $\mathfrak{G}(s, \alpha)$ is then increasing at the point and we are interested in those values of the parameter beyond the point. For example, the arc E_1 is positively strong beyond the point 1 if we assume that E_1 is in the unique continuation direction of E_{01} at 1.

From Theorem 1:2 each member of the family of extremals $x = \varphi_2(t, \sigma)$, $y = \psi_2(t, \sigma)$ is positively strong for σ sufficiently near σ_2 and $t < \tau_2(\sigma)$. Then, if $\tau_2(\sigma) > t(\sigma)$ for $\sigma \neq \sigma_2$ as described in the theorem, each extremal touches the boundary B at a point at which it is positively strong except E_2 , which ceases to be strong at the point 2. This says that the condition of the theorem is necessary. In order to prove the condition sufficient, we suppose that in every neighborhood of σ_2 there is a value of $\sigma \neq \sigma_2$ such that $\tau_2(\sigma) \leq t(\sigma)$. Choose an extremal of the family tangent to B, say for $\sigma = \sigma_4$, such that the element $[\tau_2(\sigma_4), \sigma_4, \varphi_{2t}(\tau_2(\sigma_4), \sigma_4), \psi_{2t}(\tau_2(\sigma_4), \sigma_4)]$ is in a neighborhood $N_\epsilon(\tau_2(\sigma_2), \sigma_2, \bar{x}_2', \bar{y}_2')$ and $\tau_2(\sigma_4) \leq t(\sigma_4)$, where $(\bar{x}_2', \bar{y}_2')$ determines the direction of E_{23} at 2 and $x = \varphi_3(t, \sigma)$, $y = \psi_3(t, \sigma)$ is the family of extremals (6) containing E_{23} . But

$$\mathcal{E}[\varphi_2(t, \sigma), \psi_2(t, \sigma), \varphi_{2t}(t, \sigma), \psi_{2t}(t, \sigma), \varphi_{3t}(t, \sigma), \psi_{3t}(t, \sigma)] = 0$$

for $\sigma = \sigma_4$, $t = \tau_2(\sigma_4)$ on account of the corner conditions.

Therefore,

$$\mathcal{E}[\varphi_2(t, \sigma), \psi_2(t, \sigma), \varphi_{2t}(t, \sigma), \psi_{2t}(t, \sigma), \varphi_{3t}(t, \sigma), \psi_{3t}(t, \sigma)] \leq 0$$

for $\sigma = \sigma_4$, $t = t(\sigma_4)$, since we assumed $\tau_2(\sigma_4) \leq t(\sigma_4)$. This contradicts the condition stated in the theorem.

The following modification of Theorem 2:3 will be useful later. The proof is similar to that of Theorem 2:3.

THEOREM 2:4. If $\mathcal{E}_1(\zeta_2, \eta_2, \zeta_2', \eta_2', p, q) > 0$ except for (p, q) in the direction, $(\bar{x}_2', \bar{y}_2')$, of E_{23} at 2, a necessary and sufficient condition that $\tau_2(\sigma) \geq t(\sigma)$ in a neighborhood of σ_2 is that $\mathcal{E}(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ for σ in a neighborhood of σ_2 and all $(p, q) \neq (0, 0)$.

We see that if $(\bar{x}_2', \bar{y}_2')$ is the unique continuation direction of (ζ_1', η_2') , Theorem 2:4 gives a necessary and sufficient condition in order that the families of extremals (4) and (6) may be pieced together to form a family of admissible extremals tangent to B near the point 2.

If an extremal arc ceases to be positively strong at a point for the direction $(\bar{x}', \bar{y}')$, at which the function $\Omega_0(x, y, x', y', \bar{x}', \bar{y}')$ is different from zero, then the $\mathfrak{E}$ -function becomes negative beyond this point along the extremal for certain directions.¹ This is not necessarily the case along a portion of the boundary where $T < 0$ as we have already seen at the point 2. This fact enables us to allow the function $\mathfrak{E}$ to vanish at a finite number of points and along a finite number of arcs of the boundary for certain unique directions.

Suppose that

$$\begin{aligned} \mathfrak{E}_1(\zeta, \eta, \zeta', \eta', p, q) = 0, \quad (p, q) = (h\bar{p}, h\bar{q}), \\ > 0, \quad (p, q) \neq (h\bar{p}, h\bar{q}) \quad (h > 0), \end{aligned}$$

where $\bar{p}(\sigma)$ and $\bar{q}(\sigma)$ are single-valued functions on $\sigma_4 \leq \sigma \leq \sigma_5$ of class C^1 . Suppose also that B is positively strong preceding the point 4 and following the point 5. These conditions do not deny condition IV. Then by Theorem 2:1 the conditions

$$(7) \quad \begin{aligned} F_{x'}(\zeta, \eta, \bar{p}, \bar{q}) - F_{x'}(\zeta, \eta, \zeta', \eta') &= 0, \\ F_{y'}(\zeta, \eta, \bar{p}, \bar{q}) - F_{y'}(\zeta, \eta, \zeta', \eta') &= 0 \end{aligned}$$

are satisfied for $\sigma_4 \leq \sigma \leq \sigma_5$. Since $\mathfrak{E}(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) = 0$ identically in σ on $\sigma_4 \leq \sigma \leq \sigma_5$, we have

¹This follows from the fact that $(d/ds)\mathfrak{E}(s, \bar{x}', \bar{y}') = \Omega_0(s, \bar{x}', \bar{y}')$ immediately preceding a corner and $\mathfrak{E}(s, \bar{x}', \bar{y}') = 0$ at the corner.

$$\mathcal{G}_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) = \Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) + T \sin(\theta - \phi) \equiv 0,$$

where $(\zeta', \eta') \equiv (\cos \theta, \sin \theta)$ and $(\bar{p}, \bar{q}) \equiv (\cos \phi, \sin \phi)$ just as in the proof of Theorem 2:2. If $(\bar{p}, \bar{q})$ is directed towards the left of B, then $\sin(\theta - \phi)$ is negative. In this case the function $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q})$ is negative since we have assumed that $T < 0$ along B_{12} . If $(\bar{p}, \bar{q})$ is directed towards the right of B, then $\sin(\theta - \phi)$ is positive, and in this case the function $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q})$ is positive. Thus two cases are to be considered:

(a) $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) < 0$ on $\sigma_\mu \leq \sigma \leq \sigma_\nu$. Consider the extremal $x = \varphi_2(t, \sigma_\mu)$, $y = \psi_2(t, \sigma_\mu)$ tangent to B at $t = t(\sigma_\mu)$. This extremal with the direction $(\bar{p}(\sigma_\mu), \bar{q}(\sigma_\mu))$ satisfies the corner conditions (7) at the point 4. Then the equations

$$F_{x'}(x, y, p, q) - F_{x'}(x, y, x', y') = 0,$$

$$F_{y'}(x, y, p, q) - F_{y'}(x, y, x', y') = 0,$$

where $x = \varphi_2(t, \sigma)$, $y = \psi_2(t, \sigma)$, determine functions $\tau_3(\sigma)$, $p_3(\sigma)$, and $q_3(\sigma)$ of class C^1 for $|\sigma - \sigma_\mu| \leq \epsilon$, such that

$\tau_3(\sigma_\mu) = t(\sigma_\mu)$, since $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) \neq 0$ at the point 4. By Lemma 1:2, we see that $\tau_3(\sigma) \equiv t(\sigma)$ for $\sigma_\mu \leq \sigma \leq \sigma_\nu + \epsilon$ and that $p_3(\sigma) \equiv \bar{p}(\sigma)$, $q_3(\sigma) \equiv \bar{q}(\sigma)$ on this interval. Similarly for the extremal tangent to B at the point 5 for $t = t(\sigma_\nu)$.

Therefore, the family of extremals tangent to B for

$\sigma_\mu - \epsilon \leq \sigma \leq \sigma_\nu + \epsilon$, for ϵ sufficiently small, has the corner curve

$$C_3: x = \varphi_2[\tau_3(\sigma), \sigma], \quad y = \psi_2[\tau_3(\sigma), \sigma],$$

where we define $\tau_3(\sigma) \equiv t(\sigma)$ on the interval $\sigma_\mu \leq \sigma \leq \sigma_\nu$. Also,

$\tau_3(\sigma) > t(\sigma)$ on the intervals $\sigma_f - \epsilon \leq \sigma < \sigma_f$ and $\sigma_f < \sigma \leq \sigma_f + \epsilon$ as follows from Theorem 2:3.

If we assume that the function $F_1(\zeta, \eta, \bar{p}, \bar{q})$ is not zero on the range $\sigma_f \leq \sigma \leq \sigma_f$, there will exist a family of extremals

$$(8) \quad x = \bar{\varphi}_2(t, \sigma), \quad y = \bar{\psi}_2(t, \sigma),$$

such that the direction $(\bar{\varphi}_{2t}[\tau_3(\sigma), \sigma], \bar{\psi}_{2t}[\tau_3(\sigma), \sigma])$ is the continuation direction of $(\varphi_{2t}[\tau_3(\sigma), \sigma], \psi_{2t}[\tau_3(\sigma), \sigma])$ for each σ on the range $\sigma_f - \epsilon_1 \leq \sigma \leq \sigma_f + \epsilon_1$, if ϵ_1 is sufficiently small.

THEOREM 2:5. The determinant

$$\bar{D}_2(t, \sigma) \equiv \begin{vmatrix} \bar{\varphi}_{2t} & \bar{\varphi}_{2\sigma} \\ \bar{\psi}_{2t} & \bar{\psi}_{2\sigma} \end{vmatrix}$$

does not vanish for $t = \tau_3(\sigma)$, $\sigma_f \leq \sigma \leq \sigma_f$.

Suppose for the moment that t is the length of arc along each extremal of the family (8) measured from the point of intersection with the boundary. Then we have the following identities:

$$\zeta'(\sigma) \equiv \bar{\varphi}_{2\sigma}(0, \sigma), \quad \eta'(\sigma) \equiv \bar{\psi}_{2\sigma}(0, \sigma).$$

Thus the determinant $\bar{D}_2(0, \sigma)$ does not vanish on $\sigma_f \leq \sigma \leq \sigma_f$, since the directions $(\bar{\varphi}_{2t}(0, \sigma), \bar{\psi}_{2t}(0, \sigma))$ and $(\zeta'(\sigma), \eta'(\sigma))$ do not coincide on this interval.

The other case to be considered is;

(b) $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) > 0$ on $\sigma_f \leq \sigma \leq \sigma_f$. In this case each extremal tangent to B on $\sigma_f \leq \sigma \leq \sigma_f$ commences to be positively strong at its point of tangency, as follows from Lemma 2:1. That is, these extremals are positively strong beyond the corner-curve C_3 . Therefore, each member in a neighborhood of the extremals tangent to B at the points 4 and 5 will be positively

strong beyond their corner-curve. This says that in this case each extremal tangent to B is either positively strong at its point of contact with B or commences to be positively strong there. We have no use for the family of extremals analogous to the family (8) in the preceding case, since in this case these arcs are directed to the right of B and are not admissible.

Evidently, the case where the $\mathcal{C}$ -function vanishes along a finite number of arcs and points of the boundary B for certain directions $(\bar{p}(\sigma), \bar{q}(\sigma))$ as already described for the arc B_{45} , causes no additional difficulties. This may happen along an arc of B containing the point 1 in the case where $\Omega_0 > 0$. Also, this may happen along an arc of B containing the point 2. This would simply mean that the corner curve C_2 coincides with B along an arc of B .

2:4. Analogue of the Jacobi condition. We shall determine under what conditions the usual geometric proof of the Jacobi condition, by means of the envelope theorem, can be employed here. It is evident that no difficulties present themselves in proving the envelope theorem for the family of extremals (3) containing the arc E_{01} . However, the envelope theorem for extremaloids cannot be extended to the family of extremaloids (6) containing the arc E_{23} without some additional assumption as in Theorem 2:4. This makes it desirable to seek a new theorem concerning the envelope of a one-parameter family of extremaloids which will enable us to prove the Jacobi condition along the arc E_{23} without further assumptions. Also the usual methods are not sufficient to prove a Jacobi condition for an arc of an extremal which is a part of the boundary curve. We shall develop a new method which enables us to take care of both these cases.

V₁. If the determinant

$$D_0(t, a_0) \equiv \begin{vmatrix} \phi_{0t} & \phi_{0a} \\ \psi_{0t} & \psi_{0a} \end{vmatrix} \Big|_{a=a_0}$$

vanishes on the interval $t_0 < t \leq t_1$ and if the envelope of the family (3) has a regressive branch at its corresponding point of contact with the arc E_{01} , then the arc E_{03} cannot minimize the integral I .¹

V₂. If $G_1(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ in a neighborhood of σ_2 for all $(p, q) \neq (0, 0)$ and equals zero at $\sigma = \sigma_2$ only for the unique continuation direction $(p, q) = (h\bar{x}_2', h\bar{y}_2')$; if the determinant

$$D_3(t, \sigma_2) \equiv \begin{vmatrix} \phi_{3t} & \phi_{3\sigma} \\ \psi_{3t} & \psi_{3\sigma} \end{vmatrix} \Big|_{\sigma=\sigma_2}$$

vanishes on the interval $t_2 \leq t \leq t_3$; and if the envelope of the family (6) has a regressive branch at its corresponding point of contact with the arc E_{23} ; then the curve E_{03} cannot minimize the integral I .

We know that the determinant $D_3(t_2, \sigma_2)$ is not zero by Theorem 2:5 applied at the point 2 since the corner-curve C_2 is tangent to B at 2. In order to prove this condition on the interval $t_2 < t \leq t_3$, we shall suppose that the determinant $D_3(t_7, \sigma_2)$ is zero, where t_7 is some value of t on this interval. Then the equation

$$D_3(t, \sigma) = 0$$

defines the functions $t = T(u)$, $\sigma = \Sigma(u) = \pm u$, reducing to (t_7, σ_2) for $u = u_2$ and of class C^1 near u_2 , since the derivative $D_{3t}(t_7, \sigma_2)$ is not zero.² Therefore, the equations of the

¹Bolza, Lectures, p. 204.

²Ibid., p. 200.

enveloping curve take the form

$$e: x = \varphi_3 [T(u), \Sigma(u)] \equiv X(u), \quad y = \psi_3 [T(u), \Sigma(u)] \equiv Y(u),$$

where the functions $X(u)$ and $Y(u)$ are of class C^1 near u_2 .

LEMMA 2:2. If $\mathcal{G}_1(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ in a neighborhood of σ_2 for all $(p, q) \neq (0, 0)$ and equals zero at $\sigma = \sigma_2$ only for the unique continuation direction $(p, q) = (h\bar{x}', h\bar{y}')$; if e is the envelope of the family of extremals (6) containing E_{23} and has a regressive branch at its point of contact 7 with E_{23} ; then

$$I(E_{01456} + e_{67}) = I(E_{0127}).$$

For each value of u near

u_2 there exists an extremaloid

E_{456} which is tangent to the

envelope e at the point 6 and

e is tangent to E_{23} at the

point 7. This extremaloid is

either tangent to B at the

point 4 or intersects B at an angle which makes the $\mathcal{G}$ -function

zero there, as follows from Theorem 2:4. Define the function

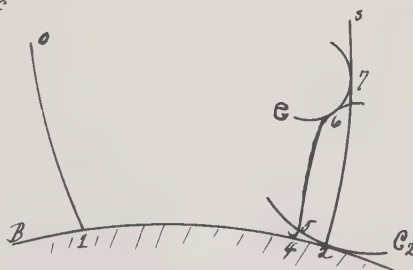
$I(u)$ as follows:

$$\begin{aligned} I(u) = & I(E_{01}) + \int_1^{\Sigma(u)} F(\zeta, \eta, \zeta', \eta') d\sigma + \int_{t[\Sigma(u)]}^{z[\Sigma(u)]} F(x, y, x', y') dt \\ & + \int_{z[\Sigma(u)]}^{T(u)} F(x, y, x', y') dt + \int_u^{u_2} F(X, Y, X', Y') du, \end{aligned}$$

where

$$(9) \quad x = \varphi_2 [t, \Sigma(u)], \quad y = \psi_2 [t, \Sigma(u)],$$

$$(10) \quad x = \varphi_3 [t, \Sigma(u)], \quad y = \psi_3 [t, \Sigma(u)],$$



and their respective derivatives are the arguments in the third and fourth terms respectively. Then the function $I(u)$ is differentiable near u_2 . Since the families (9) and (10) satisfy the hypotheses of Theorem 1:4, we have

$$I'(u) = \mathbb{E}(x, y, x', y', \zeta', \eta')^4 - \mathbb{E}(x, y, x', y', X', Y')^6 = 0$$

on $|u - u_2| \leq \epsilon$, if ϵ is sufficiently small. Thus $I(u)$ is a constant function of u on this interval, as was to be proved.

Now, the envelope e cannot be an extremal since the only extremal through the point 6 in the direction of E_{56} is E_{56} , and E_{23} is the only extremal through the point 7 in the direction of E_{23} . Therefore, $E_{01456} + e_{67}$ cannot be a part of the minimizing curve. Consequently the curve E_{03} cannot minimize the integral I on account of Lemma 2:2. This completes the proof of condition V_2 .

The condition V_2 requires the rather restrictive assumption that $\mathbb{E}_1(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ on $\sigma_2 \leq \sigma \leq \sigma_2 + h_2$. The following condition does not require this additional assumption but instead requires that the envelope be non-singular:

V_3 . If the direction $(\bar{x}_2', \bar{y}_2')$ of E_{23} is the unique continuation direction corresponding to the direction (ζ_2', η_2') of B_{12} at the point 2, and if E_{23} contains a point of contact between the points 2 and 3 with a non-singular envelope e of the family of extremals (6) containing E_{23} , then the curve E_{03} cannot minimize the integral I .

Suppose that the envelope has the equations as described in the proof of condition V_2 and has the point 7 between the points 2 and 3, in common with the arc E_{23} . Since the derivative $D_{3t}(t_7, \sigma_2)$ is different from zero and e is assumed to be non-singular at the point 7, e lies entirely on one side of E_{23} for

σ in a neighborhood of σ_2 . For,

$$D_{3t}(t, \sigma) \Big|^7 = \frac{1}{\rho} - \frac{1}{r} \Big|^7,$$

where $1/\rho$ and $1/r$ are the curvatures of e and E_{23} respectively, if we assume for the moment that the arc length is the parameter on both the curves e and E_{23} .

Since the curve E_{03} minimizes the integral I , we have $\mathfrak{C} \geq 0$ along B preceding the point 2. Hence, by Theorem 2:4, $\tau_2(\sigma) \geq t(\sigma)$ on $\sigma_2 - h_2 \leq \sigma \leq \sigma_2$ for h_2 sufficiently small. Then if e is to the left of E_{23} , the Lemma 2:2 still applies. If e is to the right of E_{23} this method fails. In this case consider the members of the family of extremals

$$(6) \quad x = \varphi_3(t, \sigma), \quad y = \psi_3(t, \sigma),$$

tangent to e for $\sigma_2 \leq \sigma \leq \sigma_2 + \epsilon$. Bolza¹ has proved that these extremals simply cover the region in the xy -plane determined by the equations (6) for

$$\sigma_2 \leq \sigma \leq \sigma_2 + \epsilon, \quad T(\sigma) - \rho \leq t \leq T(\sigma),$$

in this case if ϵ and ρ are sufficiently small. The single-valued functions $t \equiv f(x, y)$, $\sigma(x, y)$, which determine the unique extremal of the family (6) through each point of this region, are of class C^1 interior to the region and continuous on the boundary. Now consider the family of admissible curves

¹Existence proof for a field of extremals tangent to a given curve, Transactions of the American Mathematical Society, VIII (1907), pp. 399-404.

$$\begin{aligned}
 x &= \xi(\nu), & y &= \eta(\nu), & (\sigma_1 \leq \sigma \leq \sigma(\nu)), \\
 x &= \phi_2[t, \sigma(\nu)], & y &= \psi_2[t, \sigma(\nu)], & (t[\sigma(\nu)] \leq t \leq \tau_2[\sigma(\nu)]), \\
 E_{\nu}: & & & & \\
 x &= \phi_3[t, \sigma(\nu)], & y &= \psi_3[t, \sigma(\nu)], & (\tau_2[\sigma(\nu)] \leq t \leq f(\nu)), \\
 x &= \phi_3(t, \sigma_2) \equiv \hat{x}(t), & y &= \psi_3(t, \sigma_2) \equiv \hat{y}(t), & (\nu \leq t \leq t_3),
 \end{aligned}$$

where the functions $f(\nu) \equiv f[\hat{x}(\nu), \hat{y}(\nu)]$ and $\sigma(\nu) \equiv \sigma[\hat{x}(\nu), \hat{y}(\nu)]$ determine the unique extremal of the family (6) through each point of E_{23} in the interval $\nu_7 \leq \nu \leq \nu_7 + \epsilon$ for ϵ sufficiently small. The function

$$I(\nu) \equiv I(E_{01}) + I(E_{\nu})$$

is continuous on $\nu_7 \leq \nu \leq \nu_7 + \epsilon_1$ and has a derivative on $\nu_7 < \nu \leq \nu_7 + \epsilon_1$ for ϵ_1 sufficiently small. This follows from the properties of the functions $f(x, y)$ and $\sigma(x, y)$. Then by Theorem 1:4, we have

$$\begin{aligned}
 I'(\nu) &= F_{\hat{x}, \hat{x}'} + F_{\hat{y}, \hat{y}'} \left| \begin{matrix} f(\nu) \\ -F(x, y, x', y') \end{matrix} \right|_{\nu} \\
 &= -\mathcal{E}(x, y, x', y', \hat{x}', \hat{y}') \left| \begin{matrix} f(\nu) \end{matrix} \right|.
 \end{aligned}$$

Since the curve E_{03} is supposed to minimize the integral I , it follows from condition IV that $\mathcal{E}(x, y, x', y', p, q) \geq 0$ along E_{23} for all $(p, q) \neq (0, 0)$. If $F_1 > 0$ along E_{03} then $F_1(x, y, x', y') > 0$ for all elements (x, y, x', y') in a certain neighborhood of those on E_{03} . Hence $\mathcal{E}(x, y, x', y', p, q) > 0$ along each member of the family (6) sufficiently near E_{23} for all (p, q) in a sufficiently small neighborhood of (x', y') . Also, we have

$$\lim_{\nu=\nu_7} [\phi_3(t(\nu), u(\nu)), \psi_3(t(\nu), u(\nu))] = (\hat{x}', \hat{y}').$$

Therefore, it follows that

$$\mathcal{E} [\varphi_3(t, u(\nu)), \psi_3(t, u(\nu)), \varphi'_3(t, u(\nu)), \psi'_3(t, u(\nu)), \hat{x}', \hat{y}']^{t(\nu)} > 0$$

on $\nu_7 < \nu \leq \nu_7 + \epsilon_1$ for ϵ_1 sufficiently small, since the direction $[\varphi'_3(t(\nu), u(\nu)), \psi'_3(t(\nu), u(\nu))]$ does not coincide with the direction $(\hat{x}', \hat{y}')$ on this interval. Consequently, the derivative $I'(\nu)$ is negative for $\nu > \nu_7$ but sufficiently near ν_7 , and the curve E_{03} cannot minimize the integral I . This is without the assumption that $\mathcal{E} \geq 0$ along B beyond the point 2.

The next condition has to do with the behavior of the corner curve C_1 in a neighborhood of the point 1.

VI. If the direction (ξ', η') of B_{12} is the unique continuation of the direction (x', y') of E_{01} at the point 1; if E_{01} is positively strong preceding 1; if the curve E_{03} minimizes the integral I ; then the corner curve C_1 crosses E_{03} at 1 but may coincide with the boundary B along an arc following the point 1.

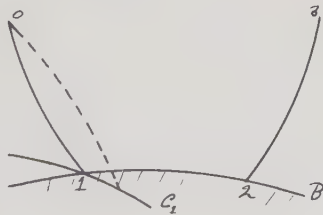
The equations

$$\xi(\sigma) = \varphi_0(t, a), \quad \eta(\sigma) = \psi_0(t, a)$$

determine functions $t(\sigma)$ and $a(\sigma)$ of class C^1 near σ_1 , since the determinant $D_0(t_1, a_0)$ is not zero and the equations have the initial solution $(t, a, \sigma) = (t_1, a_0, \sigma_1)$. Then we have the identities

$$\xi(\sigma) \equiv \varphi_0[t(\sigma), a(\sigma)], \quad \eta(\sigma) \equiv \psi_0[t(\sigma), a(\sigma)], \quad (|\sigma - \sigma_1| \leq \epsilon).$$

Suppose that the condition is not true as shown in the figure. Consider the family of admissible curves



$$E_{\sigma} : \begin{aligned} x &= \varphi_0(t, a(\sigma)), & y &= \psi_0(t, a(\sigma)) & (t_0 \leq t \leq t(\sigma)), \\ x &= \zeta(u), & y &= \eta(u) & (\sigma \leq u \leq \sigma_2). \end{aligned}$$

The function

$$I(\sigma) = \int_0^{t(\sigma)} F(x, y, x', y') dt + \int_{\sigma}^{\sigma_2} F(\zeta, \eta, \zeta', \eta') du$$

is differentiable and has the derivative

$$I'(\sigma) = -G(x, y, x', y', \zeta', \eta') \Big|_{t(\sigma)}$$

by Theorem 1:4. For $\sigma > \sigma_1$ but near σ_1 , we see geometrically that the point corresponding to $t = t(\sigma)$ precedes the curve C_1 on the extremal corresponding to $a = a(\sigma)$. But in this case

$G_1(x, y, x', y', \zeta', \eta') > 0$ by Theorem 1:2, since we have assumed that the extremal arc E_{01} is positively strong preceding the point 1. This says that $I'(\sigma) < 0$ for $\sigma > \sigma_1$ but sufficiently near σ_1 . Thus the curve E_{03} could not minimize the integral I .

The corner-curve C_1 cannot be tangent to E_{01} , since E_{01} contains no pair of conjugate points. Let us define

$$D_1(t, a) \equiv \begin{vmatrix} \varphi_{1t} & \varphi_{1a} \\ \psi_{1t} & \psi_{1a} \end{vmatrix}.$$

Then, if the determinant $D_1(t_1, a_0)$ is not zero, it has the same sign as the determinant $D_0(t_1, a_0)$. However, the determinant $D_1(t_1, a_0)$ may vanish, in which case the corner curve C_1 would at least be tangent to B_{12} at the point 1.

2:5. Modification of the original hypotheses. Up to this point we have assumed, for convenience, that the extremal arcs E_{01} and E_{23} are of class C^1 . We may assume that these curves have a finite number of corners distinct from the points 1 and 2. Then, since these arcs are interior to the region of admissible curves except at 1 and 2, we may assume the results of Graves for these

arcs with the conditions imposed by the boundary.¹ That is, there exists a family

$$(12) \quad x = \phi_0(t, a), \quad y = \psi_0(t, a),$$

of extremaloids passing through the point 0 and containing the extremaloid $E_{01} + E_1$ for $a = a_0$, $t_0 \leq t \leq t_1 + \rho$. This family has the same properties as the family (3) except for values of t at corners. Similarly, there exists a family

$$(13) \quad x = \phi_3(t, \sigma), \quad y = \psi_3(t, \sigma)$$

of extremaloids, containing the extremaloid E_{23} for $\sigma = \sigma_2$, $t_2 \leq t \leq t_3$ and having the same properties as the family of extremals (6) except for t at corners. These results depend upon the assumption that $\Omega_0(x, y, x', y', \bar{x}', \bar{y}') \neq 0$ at the corners of E_{03} interior to the admissible region. Consequently, from this point on in this chapter we shall assume that the function Ω_0 is not zero at the corners of E_{03} interior to the admissible region.

The conditions of Section 2:4 under these new assumptions are as follows:

V_1 . If the determinant $D_0(t, a_0)$ vanishes on the interval $t_0 < t \leq t_1$ except possibly the first end point of each extremal arc and if the envelope of the family (12) has a regressive branch at its corresponding point of contact with the extremaloid E_{01} , then the curve E_{03} cannot minimize the integral I .

V_2 . If $E(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ in a neighborhood of σ_2 for all $(p, q) \neq (0, 0)$ and equals zero at $\sigma = \sigma_2$ only for the unique continuation direction $(p, q) = (h\bar{x}_2', h\bar{y}_2')$; if the determinant $D_3(t, \sigma_2)$ vanishes on the interval $t_2 \leq t \leq t_3$ except

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), pp. 4-13.

possibly at the first end point of each extremal arc; and if the envelope of the family (13) has a regressive branch at its corresponding point of contact with the extremaloid E_{23} ; then the curve E_{03} cannot minimize the integral I.

The next condition has to do with the Jacobi condition on the extremaloid E_{23} without the assumption that $\mathcal{G} \geq 0$ along B beyond the point 2 which is necessary for the usual proof of condition V_2 .

V_3 . If the direction $(\bar{x}_2', \bar{y}_2')$ of E_{23} is the unique continuation direction corresponding to the direction (ξ_2', η_2') of B_{12} at the point 2; if E_{23} contains a point of contact between the end points of one of its extremal arcs with a non-singular envelope c of the family of extremaloids (13); then the curve E_{03} cannot minimize the integral I.

The end points of each extremal arc are excluded since the proof is made by combining the usual envelope theorem with the methods developed in the proof of condition V_3 of Section 2:4. The first of these methods does not hold at the first end point of each extremal arc and the second fails at the second end point.

The next three conditions concerning the corners of E_{03} parallel respectively the preceding three conditions. The proofs¹ of these conditions depend upon the additional assumption that there is a unique continuation direction at each corner.

V_{10} . If the curve E_{03} minimizes the integral I and if the extremaloid E_{01} is positively strong, then the determinant $D_0(t, a_0)$ does not change sign at the corners of the extremaloid $E_{01} + E_1$.

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 14.

V_{2C}. Suppose that the curve E_{03} minimizes the integral I and that $\mathcal{G}_1(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ in a neighborhood of σ_2 for all $(p, q) \neq (0, 0)$ and equals zero at $\sigma = \sigma_2$ only for the unique continuation direction $(p, q) = (h\bar{x}_2', h\bar{y}_2')$. Also suppose that the extremaloid E_{23} is positively strong. Then the determinant $D_3(t, \sigma_2)$ does not change sign at the corners of E_{23} .

V_{3C}. Suppose that the curve E_{03} minimizes the integral I and that E_{23} is positively strong. Also suppose that each extremal arc of E_{23} is directed to the left of the one preceding. Then the determinant $D_3(t, \sigma_2)$ does not change sign at the corners of the extremaloid E_{23} .

2:6. Existence of a field of extremaloids. It will be shown in this section that the necessary conditions already developed, when modified in the usual way, are sufficient to insure the existence of a field of extremaloids tangent to the boundary B . We shall suppose that the following conditions are satisfied:

I. E_{01} and E_{23} are extremaloids without multiple points such that $\Omega_0 \neq 0$ at the corners of E_{03} ;

II'. $T < 0$ along the arc B_{12} ;

III. $\mathcal{G}(x, y, x', y', \zeta', \eta') = 0$ at the points 1 and 2 for (x', y') in the direction of E_{01} and E_{23} respectively;

IV'. E_{01} and E_{23} are positively strong and each uniquely determined by each of its elements (x, y, x', y') ; B is positively strong on $\sigma_1 \leq \sigma \leq \sigma_2 + h_2$ except possibly for a finite number of points and arcs, at each point of which $\mathcal{G}_1(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) = 0$ for a unique direction $(\bar{p}(\sigma), \bar{q}(\sigma))$, where the functions $\bar{p}(\sigma)$ and $\bar{q}(\sigma)$ are of class C^1 on these arcs and are such that $F_1(x, y, \bar{p}, \bar{q}) \neq 0$;

V'. The determinant $D_0(t, a_0)$ does not vanish nor change sign on $t_0 < t \leq t_1$; the determinant $D_1(t_1, a_0)$ is not zero and has the sign of $D_0(t, a_0)$; and the determinant $D_3(t, \sigma_2)$ does not vanish nor change sign on the interval $t_2 \leq t \leq t_3$.

From these conditions it can be shown that there exists a family of extremaloids

$$(14) \quad x = \varphi(t, a), \quad y = \psi(t, a)$$

through a point O' preceding O on E_{01} , containing $E_{01} + E_1$ for $a = a_0$, $t_0 \leq t \leq t_1 + \rho$ and such that the determinant

$$D(t, a_0) \equiv \begin{vmatrix} \varphi_t & \varphi_a \\ \psi_t & \psi_a \end{vmatrix}_{a=a_0}$$

does not vanish on $t_0 \leq t \leq t_1 + \rho$ and does not change sign at the corners.¹

THEOREM 2:6. The extremaloids of the family (14) form a field $\mathcal{F}_1$ in the region of the xy -plane determined by $t_0 \leq t \leq t_1 + \rho$, $|a - a_0| \leq \epsilon$, for ρ and ϵ sufficiently small. The single-valued functions $t(x, y)$, $a(x, y)$, which determine the unique extremaloid through each point of $\mathcal{F}_1$, are of class C^1 in $\mathcal{F}_1$ except at corners.²

THEOREM 2:7. The extremals of the family (4) form a field $\mathcal{F}_2$ adjoining B in the region of the xy -plane determined by $t(\sigma) \leq t \leq t(\sigma) + \rho$, $\sigma_1 \leq \sigma \leq \sigma_2 + h_2$, for ρ and h_2 sufficiently small. The functions $t(x, y)$, $\sigma(x, y)$ which determine the unique extremal through each point of $\mathcal{F}_2$ are single-valued, of class C^1 in $\mathcal{F}_2$, and continuous on the boundary.

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 25.

²Ibid., p. 25.

The determinant $D_2(t, \sigma)$ has the derivative

$$D_{2t}(t, \sigma) \left[t(\sigma) = \frac{1}{\rho} \right]^\sigma - \frac{1}{r} \left] t(\sigma), \right.$$

where $1/\rho$ and $1/r$ are the curvatures of the boundary B and the member of the family (4) tangent to B at $t(\sigma)$ respectively.

Since $T < 0$ along the arc B_{12} and $T = 0$ along each member of the family tangent to B , we have that the determinant $D_{2t}(t(\sigma), \sigma)$ is less than zero on the interval $\sigma_1 \leq \sigma \leq \sigma_2 + h_2$ for h_2 sufficiently small. Bolza¹ has shown that ρ can be so chosen that the equations (4) define a one-to-one correspondence between the region $t(\sigma) \leq t \leq t(\sigma) + \rho$, $\sigma_1 \leq \sigma \leq \sigma_2 + h_2$ in the $t\sigma$ -plane and its image $\mathcal{T}_2$ in the xy -plane. For (t, σ) interior to this region, we have the existence of the functions $t(x, y)$, $\sigma(x, y)$ of class C^1 from an application of the implicit function theorems. If a sequence of points $(x, y)_n$ of the region approaches a point of the boundary B , say (x_1, y_1) , the corresponding values $(t, \sigma)_n$ defined by the equations of the theorem must approach the values (t_1, σ_1) which define (x_1, y_1) by means of equations (4), as may be seen from the continuity of the functions ϕ_2 and ψ_2 . Hence the functions $t(x, y)$ and $\sigma(x, y)$ are continuous over the entire region including the boundary. This completes the proof of the theorem.

The derivatives of the functions $t(x, y)$ and $\sigma(x, y)$ on B are in general infinite. For, by differentiating the identities

$$x = \phi_2[t(x, y), \sigma(x, y)], \quad y = \psi_2[t(x, y), \sigma(x, y)],$$

we get

$$1 = \phi_{2t} \frac{\partial t}{\partial x} + \phi_{2\sigma} \frac{\partial \sigma}{\partial x}, \quad 0 = \psi_{2t} \frac{\partial t}{\partial x} + \psi_{2\sigma} \frac{\partial \sigma}{\partial x}.$$

¹Existence proof for a field of extremals tangent to a given curve, Transactions of the A. M. S., VIII (1907), pp. 399-404.

From this it follows that

$$\frac{\partial t}{\partial x} D_2(t, \sigma) = \psi_{2\sigma}, \quad \frac{\partial \sigma}{\partial x} D_2(t, \sigma) = -\psi_{2t},$$

and similarly for the derivatives $\partial t/\partial y$ and $\partial \sigma/\partial y$.

THEOREM 2:8. The family of extremals (8) forms a field $\mathcal{F}_j$ in the region of the xy -plane determined by $\tau_j(\sigma) \leq t \leq \tau_j(\sigma) + \epsilon_1$, $\sigma_t - \epsilon_2 \leq \sigma \leq \sigma_t + \epsilon_2$, for ϵ_1 and ϵ_2 sufficiently small. The functions $t(x, y)$ and $\sigma(x, y)$ are single-valued and of class C^1 throughout $\mathcal{F}_j$.

The system of equations

$$x = \bar{\varphi}_2(t, \sigma), \quad y = \bar{\psi}_2(t, \sigma)$$

satisfies the hypotheses of Bolza's theorem on the inversion of functional systems,¹ where his region $\mathcal{G}$ is $t = \tau_j(\sigma)$, $\sigma_t \leq \sigma \leq \sigma_t$, on account of Theorem 2:5. Therefore, this system defines a one-to-one correspondence between the region $\mathcal{F}_j$ in the xy -plane and the region of the theorem in the $t\sigma$ -plane and the inverse functions $t(x, y)$, $\sigma(x, y)$ are of class C^1 in $\mathcal{F}_j$.

THEOREM 2:9. The family of extremaloids (13), containing E_{23} , forms a field $\mathcal{F}_t$ in the region of the xy -plane determined by $\tau_2(\sigma) \leq t \leq t_3 + \delta$, $|\sigma - \sigma_2| \leq h_2$, for δ and h_2 sufficiently small. The functions $t(x, y)$ and $\sigma(x, y)$ are of class C^1 and single valued in $\mathcal{F}_t$ except at corners.²

Now, let $\mathcal{F}$ denote the region of points (x, y) determined by

(1) equations (14) for:

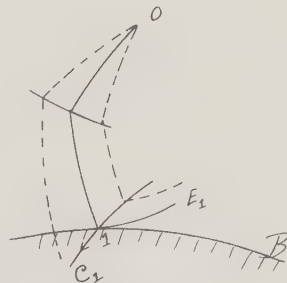
¹Bolza, Vorlesungen, p. 160.

²Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 25.

$$t_0 \leq t \leq \lambda(\sigma), \quad a_0 \leq a \leq a_0 + \epsilon,$$

$$t_0 \leq t \leq \tau_1(a) + \rho, \quad a_0 - \epsilon \leq a < a_0,$$

where $\lambda(\sigma)$ is the value of the parameter t at which the extremals meet the curve B , and where we suppose for the moment that the positive direction of C_1 is to the right of B ;



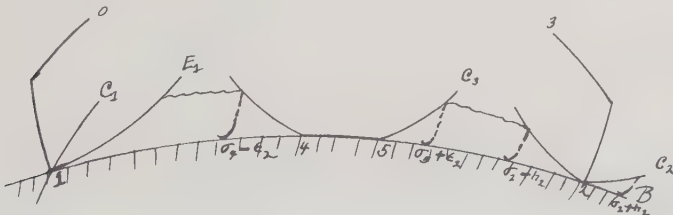
(2) equations (4) in

Case a. ($\Omega_0 < 0$ on $\sigma_1 \leq \sigma \leq \sigma_2$) for:

$$t(\sigma) \leq t \leq t(\sigma) + \rho, \quad \sigma_1 \leq \sigma \leq \sigma_1 - \epsilon_2, \quad \sigma_1 + \epsilon_2 < \sigma \leq \sigma_2 - h_2,$$

$$t(\sigma) \leq t \leq \tau_1(\sigma), \quad \sigma_1 - \epsilon_2 < \sigma \leq \sigma_1 + \epsilon_2,$$

$$t(\sigma) \leq t \leq \tau_2(\sigma), \quad \sigma_2 - h_2 < \sigma \leq \sigma_2 + h_2,$$



Case b. ($\Omega_0 > 0$ on $\sigma_1 \leq \sigma \leq \sigma_2$) for:

$$t(\sigma) \leq t \leq t(\sigma) + \rho, \quad \sigma_1 \leq \sigma \leq \sigma_2 - h_2,$$

$$t(\sigma) \leq t \leq \tau_2(\sigma), \quad \sigma_2 - h_2 < \sigma \leq \sigma_2 + h_2;$$

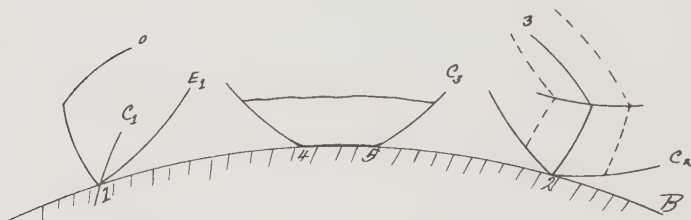


(3) equations (8) in Case a above for

$$\tau_3(\sigma) < t \leq \tau_3(\sigma) + \epsilon_1, \quad \sigma_4 - \epsilon_2 \leq \sigma \leq \sigma_5 + \epsilon_2;$$

(4) equations (13) for

$$\tau_2(\sigma) < t \leq t_3 + \delta, \quad |\sigma - \sigma_2| \leq h_2.$$



The constants ϵ , ϵ_1 , and δ may be chosen sufficiently small independently. The constants ϵ_2 , h_2 and ρ are chosen sufficiently small so that the regions of $\mathcal{F}$ connect up and do not overlap. Then the previous theorems of this section justify

THEOREM 2:10. The region $\mathcal{F}$, if sufficiently restricted, is a field of extremaloids such that through each point of $\mathcal{F}$ not on B there passes one and only one extremaloid of the field. Each point of $\mathcal{F}$ is joined to the point O' by a unique curve E of class C' except at corners, which consists either of an extremaloid which does not intersect E_{O3} or of an arc of E_{O3} plus an arc of an extremaloid which is tangent to B.

THEOREM 2:11. If $\mathcal{F}$ is sufficiently restricted, then each extremaloid of $\mathcal{F}$ is positively strong.

Each member of the family of extremaloids (12) containing $E_{01} + E_1$ is positively strong for σ sufficiently near σ_0 , as follows from Theorem 1:2. Likewise each member of the family of extremaloids (13) containing E_{23} is positively strong for σ sufficiently near σ_2 , since E_{23} is positively strong by condition IV'. Each member of the family of extremals tangent to B is positively strong for

$$t(\sigma) \leq t \leq t(\sigma) + \rho, \quad \sigma_1 < \sigma \leq \sigma_4 - \epsilon_2, \quad \sigma_5 + \epsilon_2 \leq \sigma \leq \sigma_2 - h_2,$$

if ρ is sufficiently small and if we assume for definiteness that B_{12} is positively strong except at the points 1 and 2 and along the arc B_{45} . This family is also positively strong on

$$\begin{aligned} t(\sigma) &\leq t < \tau_3(\sigma), \quad \sigma_4 - \epsilon_2 \leq \sigma < \sigma_4, \quad \sigma_5 < \sigma \leq \sigma_5 + \epsilon_2, \\ t(\sigma) &\leq t < \tau_2(\sigma), \quad \sigma_2 - h_2 \leq \sigma \leq \sigma_2 + h_2, \quad \sigma \neq \sigma_2, \end{aligned}$$

for ϵ_2 and h_2 sufficiently small in the case where $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) < 0$ on the interval $\sigma_4 \leq \sigma \leq \sigma_5$, since each member is positively strong preceding its corner curve and since B is positively strong on these arcs by condition IV'. For the case when $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) > 0$ on the arc B_{45} , consider the family of extremals

$$(4) \quad x = \varphi_2(s, \sigma), \quad y = \psi_2(s, \sigma)$$

tangent to the arc B_{45} , where s is the length of arc measured from the point of tangency. From Theorem 1:1, we have

$$\mathfrak{E}_s(0, \sigma, \cos \bar{\alpha}, \sin \bar{\alpha}) = \Omega_0(0, \sigma, \cos \bar{\alpha}, \sin \bar{\alpha}),$$

for $\sigma_4 \leq \sigma \leq \sigma_5$, where

$$\mathfrak{E}(s, \sigma, \cos \alpha, \sin \alpha) \equiv \mathfrak{E}(x, y, x', y', p, q),$$

$$\Omega_0(s, \sigma, \cos \alpha, \sin \alpha) \equiv \Omega_0(x, y, x', y', p, q)$$

along the members of the family (4), and $(\cos \alpha, \sin \alpha) \equiv (p, q)$.

Since $\Omega_0(0, \sigma, \cos \bar{\alpha}, \sin \bar{\alpha})$ is positive on the arc B_{45} , there exists a positive number ϵ_1 such that for (s, α) in the neighborhood $N_{\epsilon_1}(0, \alpha)$ and $\sigma_4 \leq \sigma \leq \sigma_5$ it is true that the function

$\mathfrak{E}(s, \sigma, \cos \alpha, \sin \alpha)$ is increasing in s . Therefore, since

$\mathfrak{E}(0, \sigma, \cos \alpha, \sin \alpha) = 0$, it follows that $\mathfrak{E}(s, \sigma, \cos \alpha, \sin \alpha)$

> 0 for $s > 0$ and α in the neighborhood N_{ϵ_1} , and $\sigma_4 \leq \sigma \leq \sigma_5$. Also, for every positive number $\epsilon \leq \epsilon_1$ and every α not in the neighborhood $N_\epsilon(\bar{\alpha})$ it is true that $\mathcal{G}_1(0, \sigma, \cos \alpha, \sin \alpha) > 0$ on B_{45} . Since the function $\mathcal{G}_1(s, \sigma, \cos \alpha, \sin \alpha)$ is continuous, it follows that for every positive number $\epsilon \leq \epsilon_1$ and $\sigma_4 \leq \sigma \leq \sigma_5$ there exists a neighborhood $N_{\epsilon_2}(0)$ such that for every $s > 0$ in N_{ϵ_2} and α not in $N_\epsilon(\bar{\alpha})$ it is true that $\mathcal{G}_2(s, \sigma, \cos \alpha, \sin \alpha) > 0$. Therefore, there exists a positive number ρ which is less than or equal to the smaller of the two positive numbers ϵ_1 and ϵ_2 such that the members of the family of extremals (4) tangent to B_{45} are positively strong on the parameter interval $t(\sigma) < t \leq t(\sigma) + \rho$. This conclusion follows from the fact that for each σ on B_{45} the regions formed by the neighborhood N_{ϵ_1} and the exterior of the neighborhood N_ϵ overlap.

It remains to prove that each member of the family (8), in the case when $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q})$ is negative along the arc B_{45} , is positively strong on $\tau_3(\sigma) < t \leq \tau_3(\sigma) + \epsilon_1$, $\sigma_4 - \epsilon_2 \leq \sigma \leq \sigma_5 + \epsilon_2$. We know that the function $\mathcal{G}(\bar{\varphi}_2, \bar{\psi}_2, \bar{\varphi}_2', \bar{\psi}_2', \bar{p}, \bar{q})$ is equal to the function $\mathcal{G}(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q})$ for $t = \tau_3(\sigma)$, $\sigma_4 \leq \sigma \leq \sigma_5$, on account of the corner conditions (7). But we have assumed that $\mathcal{G}(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) > 0$ along the arc B_{45} and for all directions $(\bar{p}, \bar{q})$ different from the directions defined by (ζ', η') and $(\bar{\varphi}_{2t}, \bar{\psi}_{2t})$. Hence, we have $\mathcal{G}_1(\bar{\varphi}_2, \bar{\psi}_2, \bar{\varphi}_2', \bar{\psi}_2', \bar{p}, \bar{q}) > 0$ for $t = \tau_3(\sigma)$, $\sigma_4 \leq \sigma \leq \sigma_5$ and all $(\bar{p}, \bar{q}) \neq (h\zeta', h\eta')$. Thus we have the identical conditions holding at each point of B_{45} along the members of the family (8) as in the case when $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) > 0$, since in the present case $\Omega_0(\bar{\varphi}_2, \bar{\psi}_2, \bar{\varphi}_2', \bar{\psi}_2', \zeta', \eta') > 0$ for $t = \tau_3(\sigma)$, $\sigma_4 \leq \sigma \leq \sigma_5$. Consequently, there exists an $\epsilon_1 > 0$ such that the members of the family (8) for $\sigma_4 - \epsilon_2 \leq \sigma \leq \sigma_5 + \epsilon_2$ are positively strong on

the parameter interval $\tau_1(\sigma) < t \leq \tau_2(\sigma) + \epsilon_1$. The proof for the existence of the ϵ_1 is the same for the arc B_{45} as that for the existence of the ρ above in the case when $\Omega_0(\xi, \eta, \xi', \eta'; \bar{p}, \bar{q})$ is positive along the arc B_{45} . For this property near the points 4 and 5, apply Theorem 1:2.

2:7. Sufficient conditions for a minimum. We wish to show now that the conditions described in the preceding section are sufficient in order that the curve E_{03} minimize the integral I. This will follow from a more general sufficiency theorem.

THEOREM 2:12. Suppose that the curve E_{03} is "imbedded" in a field $\mathcal{F}$ of extremaloids adjoined to B on the left such that for each point (x, y) in $\mathcal{F}$ there exists a unique curve E satisfying the following conditions:

- (1) E joins the point O' , preceding O on E_{01} , to (x, y) ;
- (2) E is of class C^1 at least except at corners;
- (3) E is composed of an extremaloid not intersecting E_{03} , or of an arc of E_{03} plus an extremaloid tangent to B;
- (4) each extremaloid arc of E is positively strong.

Then if V is an admissible curve lying in $\mathcal{F}$ distinct from E_{03} , we must have

$$I(V_{03}) > I(E_{03}).$$

Consider the extremaloid integral

$$W(x, y) = \int_{O'}^{t(x, y)} F(x, y, x', y') dt. *$$

Except on the corner curves, the extremal arc E_1 , and the boundary B, this function W has the partial derivatives

$$W_x = F_x(x, y, p, q), \quad W_y = F_y(x, y, p, q),$$

*See Bolza, Vorlesungen, pp. 405-7.

where (p, q) are the direction components of the extremal of the field through (x, y) . These partial derivatives are continuous on the corner curves and E_1 , and approach continuous limits on the boundary B which is assumed to be of class C'' . From this it can be shown that the function $W(x, y)$ has the above derivatives throughout $\mathfrak{J}$. Suppose we prove this for the boundary B . By transforming to the normal coordinate system along B , we see that we may consider B to be a piece of the x -axis, with the field $\mathfrak{J}$ above. Let S be a bounded open region adjoining B , outside $\mathfrak{J}$, simply covered by the normals to B , having only B as the portion of its boundary in common with the boundary of $\mathfrak{J}$. Let R be a portion of $\mathfrak{J}$ simply covered by the normals to B , such that the piece of each normal in R , measured from B , has length greater than or equal to a positive number M . Also let $y_R(x)$ be the least upper bound of y in R for given x and $y_S(x)$ the least upper bound of $-y$ in S for given x . Then $y_R \geq M > 0$, and consequently $y_S(x)/y_R(x) \leq h$, for properly chosen h . Now define the function $G(x, y)$ as follows:

$$\begin{aligned} G(x, y) &\equiv W(x, y) \quad \text{for } (x, y) \text{ in } R, \\ &\equiv (1 + h)W(x, 0) - hW(x, -y/h) \quad \text{for } (x, -y/h) \text{ in } R. \end{aligned}$$

Therefore, $G(x, y)$ is continuous on the region $R + B + S$. By the theorem of the mean for functions of one variable there exists a derivative

$$G_y(x, 0^+) = W_y(x, 0^+).$$

By computation there exists a derivative

$$G_y(x, 0^-) = W_y(x, 0^+).$$

Consequently, the function $W(x, y)$ has the continuous derivative

$W_y(x, y)$ even on the boundary B . Now, by the theorem of the mean, we have

$$\frac{W(\bar{x}, 0) - W(x, 0)}{\bar{x} - x} = W_x(\zeta_y, y) + \frac{W(\bar{x}, 0) - W(\bar{x}, y) + W(x, y) - W(x, 0)}{\bar{x} - x},$$

where ζ_y lies between x and $\bar{x}$. There exists a sequence $\{y_n\}$ with limit zero and such that the sequence $\{\zeta_{y_n}\}$ converges to ζ , on the closed interval $(x, \bar{x})$. Therefore, we have

$$\frac{W(\bar{x}, 0) - W(x, 0)}{\bar{x} - x} = W_x(\zeta, 0).$$

Let $\bar{x}$ approach x and then $W_x(\zeta, 0)$ approaches $W_x(x, 0)$. For $y < 0$, we have

$$G_x(x, y) = (1 + h)W_x(x, 0) - hW_x(x, -y/h).$$

It follows from these results that the function $W(x, y)$ has the continuous derivative $W_x(x, y)$ on the boundary B . The proof that $W(x, y)$ has the continuous derivatives $W_x(x, y)$ and $W_y(x, y)$ on corner curves and the extremal arc E_1 is the same as for B .¹

In order to complete the proof of the theorem, let V_{03} represent any admissible comparison curve in $\mathcal{J}$ with equations of the form

$$V_{03}: \quad x = X(v), \quad y = Y(v), \quad (v_0 \leq v \leq v_1).$$

Then, from the results of the preceding paragraph it follows that

$$W(v) = W[X(v), Y(v)] + \int_v^{v_1} F(X, Y, X', Y') dt$$

has a derivative

$$W'(v) = - \mathcal{G}(x, y, p, q, X', Y'),$$

¹This proof was suggested to me by Graves.

except at the corners of V_{03} . This says that the function $W(\nu)$ is decreasing on the interval (ν_0, ν_3) . Therefore,

$$\Delta I \equiv I(V_{03}) - I(E_{03}) = W(\nu_0) - W(\nu_3) > 0$$

unless $W(\nu)$ is identically zero on the interval (ν_0, ν_3) . In this case the curves E_{03} and V_{03} coincide throughout.¹

THEOREM 2:13. If the conditions I, II', III, IV', and V' of Section 2:6 are satisfied along the curve E_{03} , then $I(V_{03}) > I(E_{03})$ for every admissible curve V sufficiently near E_{03} but distinct from E_{03} .

This follows immediately from the general sufficiency Theorem 2:12, since it was shown in Section 2:6 that these conditions insured the existence of a field $\mathfrak{F}$ having the properties required in the sufficiency theorem.

2:8. The case when the arc B_{12} of the boundary is an extremal. The treatment of the Jacobi condition and the sufficient conditions of the preceding sections of this chapter depends essentially upon the assumption that $T < 0$ along the arc B_{12} of the boundary. We shall make use here of the method developed in Section 2:4 to prove the Jacobi condition for an arc of the boundary along which $T = 0$. We shall assume that there are no corners on the arc B_{12} of the boundary. The case when the boundary has corners will be discussed in the next section.

If we assume that the function $\Omega_0 \neq 0$ at the corners of the curve E_{03} , since $T = 0$ along each arc of E_{03} between corners,² there will exist a family of extremaloids

$$(15) \quad x = \phi(t, a), \quad y = \psi(t, a),$$

¹Bliss, Sufficient conditions for a minimum with respect to one-sided variations, Transactions A. M. S., V (1904), p. 491.

²Compare Theorem 2:2.

through the point 0 and containing E_{03} for $a = a_0$, $t_0 \leq t \leq t_3$. The functions φ , φ' , φ'' , ψ , ψ' , ψ'' are of class C^1 on $|a - a_0| \leq \epsilon$, $t_0 - \delta \leq t \leq t_3 + \delta$ except for the parameter t at corners.¹

THEOREM 2:14. If the curve E_{03} minimizes the integral I , then E_{03} contains no point of contact with a non-singular envelope of the family of extremaloids (15) between the end points of each extremal arc on $t_1 < t < t_3$.

The proof of this theorem is exactly the same as that for the condition V_3 of Section 2:4 on the arc E_{23} . The fact that the corner curves at the corners of E_{03} preceding and including the point 2 cross E_{03} is proved here geometrically just as in the case of free variations.² However, the proof fails for those corner curves at corners of the extremaloid E_{23} beyond the point 2, unless each extremal arc is directed towards the left of the one preceding.³

The set of sufficient conditions for extremaloids proved by Graves⁴ applies in this case. For, the fact that our admissible comparison curves all lie entirely on the left of the extremal arc B_{12} of the boundary makes no difference in the construction of the field of extremaloids and the invariant property of the Hilbert integral.

It is interesting to notice that the entire discussion of this case is independent of the boundary preceding the point 1

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 6.

²Ibid., p. 14.

³See V_{3C} of Section 2:5.

⁴Ibid., pp. 24-8.

and beyond the point 2. Thus we may restrict the admissible region as we please preceding 1 and following 2, so long as our boundary curve does not cross the curves E_{01} and E_{23} other than at the points 1 and 2. In fact the admissible region may be on both sides of the arc E_{12} , in which case there is no boundary preceding 1 and following 2. In this extreme case all admissible comparison curves lie entirely on the left of E_{12} or entirely on the right in a neighborhood of E_{12} .

It is also interesting to notice that the case considered in this section takes care of the case excluded by Bliss in his treatment of one-sided variations in the plane for the class of regular problems.¹

2:9. The case when the arc E_{12} of the boundary has a finite number of corners. In the preceding sections of this chapter we have assumed that the arc B_{12} of the boundary curve plus an extension at each end is of class C^n . We shall consider briefly here the case where the boundary is continuous and composed of a finite number of arcs each of class C^n . The treatment will be broken into two cases. In each case we shall assume, for convenience, that the curve B_{12} has just one corner, at the point 4 say, and that each of the two arcs B_{14} and B_{42} of the boundary has $T < 0$ along it or else has $T = 0$ along it, including the limiting values at the ends. The first case is as follows:

(A) The arc B_{42} lies to the left of the arc B_{14} in a neighborhood of the point 4. Let

$$\begin{array}{lll} E_{14}: & x = \zeta(\sigma), & y = \eta(\sigma) & (\sigma_1 \leq \sigma \leq \sigma_4), \\ E_{42}: & x = \zeta(\sigma), & y = \eta(\sigma) & (\sigma_4 \leq \sigma \leq \sigma_2), \end{array}$$

¹Bliss, Sufficient conditions for a minimum with respect to one-sided variations, Transactions of the American Mathematical Society, V (1904), pp. 477-92.

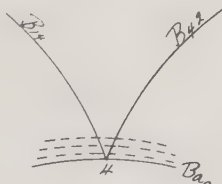
be the equations of the arcs B_{14} and B_{42} respectively.

THEOREM 2:15. If the curve E_{03} minimizes the integral I,
then $\mathfrak{E}(\zeta, \eta, \zeta', \eta', \bar{\zeta}', \bar{\eta}') = 0$ at the point 4.

Let

$$B_a: x = \varphi(t, a), \quad y = \psi(t, a),$$

be a family of curves intersecting B_{14} for $t = t_1(a)$, $\sigma = \sigma(a)$ and intersecting B_{42} for $t = t_2(a)$, $\sigma = \bar{\sigma}(a)$, where B_{a_0} is the member of this family which passes through the point 4. Then by means of the identities



$$\begin{aligned} \zeta[\sigma(a)] &\equiv \varphi[t_1(a), a], & \bar{\zeta}[\bar{\sigma}(a)] &\equiv \varphi[t_2(a), a], \\ \eta[\sigma(a)] &\equiv \psi[t_1(a), a], & \bar{\eta}[\bar{\sigma}(a)] &\equiv \psi[t_2(a), a] \end{aligned}$$

and their first derivatives with respect to a , we find that the function

$$\begin{aligned} I(a) = & \int_1^{\sigma(a)} F(\zeta, \eta, \zeta', \eta') d\sigma + \int_{t_1(a)}^{t_2(a)} F(\varphi, \psi, \varphi', \psi') dt \\ & + \int_{\bar{\sigma}(a)}^2 F(\zeta, \eta, \bar{\zeta}', \bar{\eta}') d\sigma, \end{aligned}$$

has the derivative

$$\begin{aligned} I'(a) = & \sigma' \mathfrak{E}(\varphi, \psi, \varphi', \psi', \zeta', \eta')^{t_1} - \bar{\sigma}' \mathfrak{E}(\varphi, \psi, \varphi', \psi', \bar{\zeta}', \bar{\eta}')^{t_2} \\ & - \left[F_{x_1}(\varphi, \psi, \varphi', \psi') \varphi_a + F_{y_1}(\varphi, \psi, \varphi', \psi') \psi_a \right]_1^2 \\ & + \int_{t_1}^{t_2} \frac{d}{da} F(\varphi, \psi, \varphi', \psi') dt. \end{aligned}$$

Therefore,

$$I'(a_0) = \sigma' \mathbb{E}(\varphi, \psi, \varphi', \psi', \zeta', \eta') - \bar{\sigma}' \mathbb{E}(\varphi, \psi, \varphi', \psi', \bar{\zeta}', \bar{\eta}')^4,$$

since $t_1(a) = t_2(a)$ at $a = a_0$. Now suppose that the function $\sigma(a)$ increases with a . We may choose the parameter σ so that $\sigma' = 1$ and $\bar{\sigma}' = -1$. Then we have

$$I'(a_0) = \mathbb{E}(\varphi, \psi, \varphi', \psi', \zeta', \eta') + \mathbb{E}(\varphi, \psi, \varphi', \psi', \bar{\zeta}', \bar{\eta}')^4 \leq 0,$$

since E_{03} is supposed to minimize the integral I . Let the direction of the member B_{a_0} approach the direction of B_{14} at the point 4. Then the function $\mathbb{E}(\zeta, \eta, \zeta', \eta', \bar{\zeta}', \bar{\eta}')$ is less than or equal to zero at 4. Thus $\mathbb{E}(\zeta, \eta, \zeta', \eta', \bar{\zeta}', \bar{\eta}')$ equals zero at 4, since $\mathbb{E}(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ along B_{12} for all $(p, q) \neq (0, 0)$ by condition IV.

COROLLARY. The corner conditions

$$F_x(\zeta, \eta, \bar{\zeta}', \bar{\eta}') - F_x(\zeta, \eta, \zeta', \eta') = 0,$$

$$F_y(\zeta, \eta, \bar{\zeta}', \bar{\eta}') - F_y(\zeta, \eta, \zeta', \eta') = 0$$

are satisfied at the point 4.

This follows directly from Theorem 1:5.

THEOREM 2:16. If $T < 0$ along either B_{14} or B_{42} and their directions are distinct at 4, then the function $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{\zeta}', \bar{\eta}')$ is less than zero at the point 4.

The proof of this theorem is the same as that of Theorem 2:2. This theorem excludes the case where $T = 0$ along both the arcs B_{14} and B_{42} . Therefore, if we allow corners to occur on the boundary B_{12} in the case considered in the preceding section, we must assume $\Omega_0 \neq 0$ at these corners in order to insure the existence of a corner curve C_4 determined by the family of extremaloids containing E_{04} . In this case Theorem 2:14 applies to the extremaloid E_{03} . Also, the set of sufficient conditions

for extremaloids proved by Graves¹ applies in this case.

If $T < 0$ along the arc B_{14} , there exists a corner curve C_4 tangent to B_{14} at the point 4, which is determined by the family of extremals tangent to B_{14} for values of the parameter along members of the family beyond or coinciding with that at which they are tangent to B_{14} , since $G(\xi, \eta, \xi', \eta', p, q) \geq 0$ preceding 4. If $T = 0$ along the arc B_{42} , we have a situation at the point 4 similar to that at the point 2, i.e., there exists a one-parameter family of extremals containing B_{42} and having the corner curve C_4 . If $T < 0$ along B_{42} , there exists a family of extremals tangent to B_{42} and therefore an extremal tangent to B_{42} at the point 4 in the continuation direction of B_{14} , which is analogous to the extremal E_1 at the point 1. If $T = 0$ along B_{14} and $T < 0$ along B_{42} , then we have a situation at the point 4 perfectly analogous to that at the point 1.

Finally, we consider the case excluded by the preceding theorem, i.e., where the direction of the arc B_{14} coincides with that of the arc B_{42} at the point 4. If $T = 0$ along both arcs of B_{12} , we simply have the case considered in the preceding section, since there exists only one extremal through the point 4 in the direction of B_{14} . If $T < 0$ along either of the arcs B_{14} and B_{42} , we simply have a special case of the situations already discussed in this section.

If we assume that an arc of the boundary which is an extremal is positively strong except at the ends, the proof of the general sufficiency Theorem 2:12 holds. This completes the discussion of all the possible situations in case (A) of this section.

¹Discontinuous solutions, American Journal of Mathematics, LIII (1930), pp. 24-8.

The second case to be considered here is as follows:

(B) The arc B_{42} lies to the right of the arc B_{14} in a neighborhood of the point 4. The treatment of this case is divided into two sub-cases. The first sub-case is as follows:

a. The corner conditions (2) are not satisfied along B_{12} at the point 4 for any direction. Since we have assumed that the arc B_{14} is of class C^n and that T retains its prescribed sign even at the point 4, this arc may be slightly extended beyond the point 4 in such a way that the sign of T is retained on this extension, say B_{45} . Whether T is less than zero or equal to zero along the arc B_{15} there exists a field to the left of this arc,¹ since if $T = 0$ along it we can apply the condition of Theorem 2:14 strengthened in the usual way. This field is deter-



mined by the family of extremals tangent to B_{15} beyond their point of contact with B_{15} if $T < 0$ along it and by the family of extremaloids through the point O' preceding the point O on E_{01} if $T = 0$ along B_{15} . Since we have assumed in this sub-case that the corner conditions (2) are not satisfied at the point 4 along B_{12} for any direction, it follows from condition IV and Theorem 1:5 that $\mathcal{G}_1(x, y, x', y', p, q) > 0$ along B_{12} at 4 for all $(p, q) \neq (0, 0)$. Then, if $T < 0$ along B_{15} the set of sufficient conditions of Theorem 2:13 apply to the curve E_{05} , and if $T = 0$ along B_{15} the set of sufficient conditions proved by Graves²

¹See Theorem 2:7 and its proof.

²Discontinuous solutions, American Journal of Mathematics, LII (1930), pp. 24-8.

apply to the curve E_{05} . In either case, we have $I(E_{01} + B_{15}) < I(V_{05})$, where V is any admissible comparison curve sufficiently near the curve E_{05} but distinct from it.

If $T < 0$ along the arc B_{42} , we have the existence of a field adjoining this arc on the left determined by the family of extremals tangent to it following their point of contact. Also, there is a neighborhood of the point 4 simply covered by the extremals through this point, if we assume that $F_1(x, y, p, q) \neq 0$ at 4 for all $(p, q) \neq (0, 0)$.¹ Consequently, in this case we can apply the conditions of Theorem 2.13 to the curve E_{43} and obtain the inequality $I(B_{42} + E_{23}) < I(B_{45} + V_{53})$. If $T = 0$ along B_{42} , the extremaloid E_{43} is embedded in a one-parameter family of extremaloids through the point 3, by means of which we can prove the condition of Theorem 2.14 for the curve E_{43} . In this case we can apply the set of conditions given by Graves² to the curve E_{43} and obtain the inequality $I(B_{42} + E_{23}) < I(B_{45} + V_{53})$. Combining the two inequalities $I(E_{01} + B_{15}) < I(V_{05})$ and $I(B_{42} + E_{23}) < I(B_{45} + V_{53})$, we obtain $I(E_{03}) < I(V_{03})$.

The second and last sub-case to be considered here is as follows:

b. The corner conditions (2) are satisfied by the arc B_{14} at the point 4 for some unique direction $(\bar{p}, \bar{q})$ or by the arc B_{42} at the point 4 for some unique direction $(\tilde{p}, \tilde{q})$. Suppose that the arc B_{14} satisfies the following conditions:

- (1) $T < 0$ along B_{14} ;
- (2) the direction $(\bar{p}, \bar{q})$ is to the left of the arc B_{14}

¹Bliss and Mason, Fields of extremals in space, Transactions of the American Mathematical Society, XI (1910), p. 328.

²Discontinuous solutions, American Journal of Mathematics, LII (1930), pp. 24-8.

(consequently $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) < 0$ at 4), or else the direction $(\bar{p}, \bar{q})$ is to the right of B_{14} and $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) > 0$ at 4. This states that we have the same situation along B_{14} near the point 4 as was described in the latter part of Section 2:3. Consequently, the proof of the relation $I(E_{01} + B_{15}) < I(V_{05})$ is no different from that in the preceding sub-case, provided we assume that condition IV' holds on the extended arc B_{15} . It should be noted here that this does not include the case where $(\bar{p}, \bar{q})$ is to the right of B_{14} at 4 and $\Omega_0(\zeta, \eta, \zeta', \eta', \bar{p}, \bar{q}) < 0$ at 4.

THEOREM 2:17. If $T = 0$ along the arc B_{14} and the corner curve at the point 4 of the family of extremaloids through the point 0 crosses the extremaloid $B_{14} + E_4$, where E_4 is the continuation extremal of B_{14} , then the direction $(\bar{p}, \bar{q})$ is not directed towards the exterior of the admissible region.

An indirect proof of this theorem can be made by means of a construction similar to that in the proof of condition VI of Section 2:4. This construction depends on the assumption that the functional determinant for t and a of the family of continuation extremals containing the extremal E_4 is different from zero at the point 4.

In the case covered by this theorem we cannot say that $I(E_{01} + B_{15}) < I(V_{05})$ since the extended arc B_{15} is not positively strong beyond the point 4. However, if we consider the arc B_{42} slightly extended backward to a point 6, it can be shown that $I(E_{04}) < I(V_{06} + B_{64})$ as in the preceding sub-case.

THEOREM 2:18. If $T < 0$ along the arc B_{42} and



$\Omega_0(\zeta, \eta, \zeta', \eta', \tilde{p}, \tilde{q}) < 0$ at the point 4 along B_{42} , then the direction $(\tilde{p}, \tilde{q})$ is to the left of B_{42} .

The theorem follows immediately from the inequality

$$\mathcal{E}_0(\zeta, \eta, \zeta', \eta', \tilde{p}, \tilde{q}) = \Omega_0(\zeta, \eta, \zeta', \eta', \tilde{p}, \tilde{q}) + T \sin(\theta - \phi) \geq 0$$

at the point 4, where $(\cos \theta, \sin \theta) \equiv (\zeta', \eta')$ and $(\cos \phi, \sin \phi) \equiv (\tilde{p}, \tilde{q})$.

In the case of this theorem each member of the family of extremals tangent to B_{42} is surely positively strong preceding its point of contact with B_{42} . This will also be true on the extension B_{64} if we assume that the condition IV' holds on the extended arc B_{62} . In the former case it follows just as in the preceding sub-case a that $I(E_{43}) < I(B_{45} + V_{53})$. In the latter case it follows that $I(E_{63}) < I(V_{63})$. Then, combining the inequalities $I(E_{01} + B_{15}) < I(V_{05})$ and $I(E_{43}) < I(B_{45} + V_{53})$, we conclude that $I(E_{03}) < I(V_{03})$. Similarly the two inequalities $I(E_{04}) < I(V_{06} + B_{64})$ and $I(E_{63}) < I(V_{63})$, give the inequality $I(E_{03}) < I(V_{03})$.

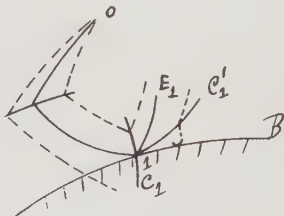
Finally, if $T < 0$ along B_{42} ; the direction $(\tilde{p}, \tilde{q})$ is towards the right of B_{42} ; and $\Omega_0(\zeta, \eta, \zeta', \eta', \tilde{p}, \tilde{q}) > 0$ along B_{42} at 4; then it follows as before that $I(E_{43}) < I(B_{45} + V_{53})$ and $I(E_{63}) < I(V_{63})$, provided we assume that condition IV' holds on the extended curve E_{63} . For, then the extremals tangent to B_{62} commence to be positively strong at the point of tangency.

It should be noted here that we have not exhausted all the possible situations at the point 4 along B_{42} for this sub-case b. For example, we have not considered the case where the direction $(\tilde{p}, \tilde{q})$ is to the left of B_{42} and $\Omega_0(\zeta, \eta, \zeta', \eta', \tilde{p}, \tilde{q}) > 0$ at 4 along B_{42} .

2:10. The extremaloid E_{01} tangent to B_{12} at the point 1 preceding the corner. In all the preceding discussion of this chapter, except the proof of Theorem 2:12, we have tacitly assumed that the extremaloid $E_{01} + E_1$ is not tangent to the arc B_{12} at 1 preceding the corner. We shall consider the case when E_{01} is tangent to B_{12} . We suppose in addition that E_{01} ceases to be positively strong at 1 for some unique direction $(\bar{x}', \bar{y}')$; that $T < 0$ along B_{12} ; and that $\Omega_0(x, y, x', y', \bar{x}', \bar{y}') \neq 0$ at 1. Then there exists a corner curve C_1 determined by the family of extremaloids

$$(12) \quad x = \varphi_0(t, a), \quad y = \psi_0(t, a)$$

through the point 0. If we assume that $F_1(x, y, \bar{x}', \bar{y}') \neq 0$ at the point 1, there exists a family of extremals



$$(5) \quad x = \varphi_1(t, a), \quad y = \psi_1(t, a),$$

which is the continuation family of (12). The member of this family through the point 1 we shall denote by E_1 as before.

THEOREM 2:19. If E_{03} minimizes the integral I ; if E_{01} is tangent to B_{12} at the point 1 and ceases to be positively strong at 1 for the unique direction $(\bar{x}_1', \bar{y}_1')$; if the corner curve C_1 crosses the extremaloid $E_{01} + E_1$ at 1 without being tangent; then the continuation direction $(\bar{x}_1', \bar{y}_1')$ is towards the interior of the admissible region.

The proof of this theorem is made indirectly by a simple geometric construction similar to that used in the proof of condition VI. We shall restrict our discussion in this section to the case of this theorem.

The family of extremals

$$(4) \quad x = \Phi_2(t, \sigma), \quad y = \Psi_2(t, \sigma)$$

tangent to the boundary for $t = t(\sigma)$ determine, in a neighborhood of the point 1, a corner curve C_1' through 1 tangent to the boundary at 1. Since $\mathcal{E}(\zeta, \eta, \zeta', \eta', p, q) \geq 0$ along the arc B_{12} by condition IV, we know by Theorem 2:4 that the points of the corner curve C_1' do not precede the points of B_{12} along the members of the family (4) in a right-hand neighborhood of σ_1 . The extremal E_1 is also a member of a family of extremals

$$(8) \quad x = \bar{\Phi}_2(t, \sigma), \quad y = \bar{\Psi}_2(t, \sigma),$$

for $\tau_1(\sigma) \leq t \leq \tau_1(\sigma) + \delta$, $\sigma = \sigma_1$, which is the continuation family of (4). The determinant $\bar{D}_2(t_1, \sigma_1)$ is not zero for this family since 1 is a focal point of the family tangent to B_{12} .

We see that the extremal E_1 is a member of the two families each having a corner curve through the point 1. Therefore, if we disregard those members of the family (5) to the right of E_1 and those members of the family (4) to the left of E_1 , we have E_1 in a region simply covered on the right of E_1 by the family (4) and its continuation (8), and on the left by those of the family (5). Consequently, the proof of the general sufficiency Theorem 2:12 applies.

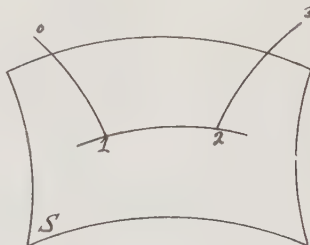
It may finally be remarked that evidently the entire development of this chapter can be applied to the case when there are a finite number of distinct segments of the minimizing curve E_{03} in common with the boundary.

PART III
UNILATERAL VARIATIONS IN THREE DIMENSIONS

3:1. Statement of the problem. Let the curve E_{03} joining the two fixed points 0 and 3, whose minimizing properties are to be studied, have equations of the form

$$x = x(s),$$

where s is the length of arc measured from the point 0. The curve E_{03} is supposed not to intersect itself; to be on one side of a given surface S ; and to be composed of three arcs E_{01} , E_{12} and E_{23} each of class C^1 . The surface S on which the arc E_{12} lies has equations of the form



$$S: x = \zeta(\alpha, \beta).$$

It has no points in common with the arcs E_{01} and E_{23} except the points 1 and 2 respectively and is further supposed to have no multiple or singular points and to be of class C''' in the vicinity of the arc E_{12} .

The function F in the integral

$$(1) \quad I = \int F(x, x') ds$$

is assumed to have the following property in addition to those enumerated at the beginning of Part I:

(C) $F_1(x, x') \neq 0$ along the curve E_{03} including both sides of corners.

The class of admissible curves is composed of those curves E which satisfy the following conditions:

(a) E passes through the fixed points 0 and 3;

(b) E lies on the same side of the boundary surface S as the arcs E_{01} and E_{23} ;

(c) E is of class D^1 ;

(d) each element (x, x') of E is in the region R^1 .

We shall combine the methods and results of the preceding chapter of this paper with those of Bliss and Underhill¹ for the regular problem.

3:2. Necessary conditions for a minimum already known.

We shall suppose that the curve E_{03} minimizes the integral I and wish to formulate its properties. We shall state here the necessary conditions for a minimum already known in order to give them a notation in keeping with that of this paper.

I. If the curve E_{03} minimizes the integral I , then the arcs E_{01} and E_{23} are arcs of space extremals.²

II. If E_{03} minimizes I , then along the arc E_{12} the equations

$$I^{(x_1)}/x_1 = \lambda(s) \quad (i = 1, 2, 3)$$

are satisfied, where (X) stands for the direction cosines of the normal of the surface S in the direction towards the side of S on which the arcs E_{01} and E_{23} lie, while $\lambda(s)$ is merely the notation

¹Transactions of the American Mathematical Society, XV (1914), pp. 291-310.

²Mason and Bliss, Properties of curves in space which minimize a definite integral, Transactions of the American Mathematical Society, IX (1908), p. 443. We shall refer to this paper as Mason and Bliss.

for the function of s defined by the three ratios.¹

III. If E_{03} minimizes I , then along E_{12} the inequality $\lambda(s) \geq 0$ is satisfied.²

IV. If E_{03} minimizes I , then $\mathcal{G}(x, x', p) \geq 0$ for all (x, x') on E_{03} and all $(p) \neq (0)$.³

V. If E_{03} minimizes I , then at the points 1 and 2 the condition $\mathcal{G}(x, x', \zeta') = 0$ is satisfied, where the direction (ζ') coincides with that of E_{12} at the points 1 and 2 respectively.⁴

In the case of the regular problem, this condition implies that the two arcs E_{01} and E_{23} are tangent to the arc E_{12} at the points 1 and 2 respectively. This is the case considered by Bliss and Underhill already referred to.

VI₁. If E_{03} minimizes I , then the arc E_{01} contains no point conjugate to the point O .⁵

3:3. Families of extremaloids associated with E_{03} . We shall show in this section the existence of certain families of extremals which may, under appropriate hypotheses, be pieced together to form families of admissible extremaloids tangent to the boundary surface S .

In the plane case we saw that the usual corner conditions were satisfied at the points 1 and 2 on account of the two conditions analogous to IV and V. We also saw that the function Ω_0 is not zero along the curve E_{03} at these points if the arcs E_{01}

¹Hadamard, Leçons sur le calcul des variations, Paris, 1910, p. 133.

²Ibid., p. 179.

³Graves, A proof of the Weierstrass condition in the calculus of variations, to be published in the American Mathematical Monthly.

⁴Hadamard, loc. cit., p. 187.

⁵Mason and Bliss, loc. cit., p. 453.

and E_{23} are not tangent to the boundary. The next two theorems show that in general these properties hold in this case.

THEOREM 3:1. If the conditions IV and V are satisfied at the points 1 and 2, then the conditions

$$(2) \quad F_{x_1}(x, \zeta') - F_{x_1}(x, x') = 0$$

are satisfied at these points of E_{03} .

This follows immediately from Theorem 1:5.

THEOREM 3:2. If $\lambda(s) > 0$ along the arc E_{12} and $x_1'X_1 \neq 0$ at the points 1 and 2 for (x') in the direction of E_{01} and E_{23} respectively, then the function $\Omega_0(\zeta, \zeta', x')$ is greater than zero at 1 and less than zero at 2.

Consider the function

$$\mathcal{G}(\zeta, \zeta', x') \equiv F(\zeta, x') - x_1'F_{x_1}F_{x_1}(\zeta, \zeta')$$

as a function of the arc length s along the arc E_{12} . From the conditions IV and V we see that this function has the derivative

$$\begin{aligned} \mathcal{G}_s(\zeta, \zeta', x') &= \zeta_1'F_{x_1}(\zeta, x') - x_1'[F_{x_1}(\zeta, \zeta') - \lambda(s)x_1] \\ &= \Omega_0(\zeta, \zeta', x') + \lambda(s)x_1'X_1 \\ &\geq 0 \end{aligned}$$

at the point 1 for (x') in the direction of the arc E_{01} there. Under the hypotheses of the theorem we see that $\Omega_0 > 0$ at the point 1 since the angle between the directions (x) and (x') at 1 is obtuse and consequently $x_1'X_1 < 0$. Similarly, $\Omega_0 < 0$ at the point 2 since $\mathcal{G}_s(\zeta, \zeta', x') \leq 0$ and $x_1'X_1 > 0$ there.

If $x_1'X_1 = 0$ at the point 1, say, then the arc E_{01} is tangent to the surface S and Theorem 3:2 does not apply. If E_{01} is tangent to the arc E_{12} on S , then $\Omega_0 = 0$ at 1. In view of these facts, we shall assume from this point on that

$\Omega_0(\zeta, \zeta', x') \neq 0$ at the points 1 and 2 when the directions (ζ') and (x') are distinct, and that $\lambda(s) > 0$ along the arc E_{12} .

The totality of solutions

$$(3) \quad x = \varphi(s, a, b)$$

of the Euler differential equations passing through the point 0 contains two arbitrary constants. The extremal E_{01} is contained in this set for certain parameter values $a = a_0$, $b = b_0$, and the functions φ , φ_s are of class C^1 for $0 \leq s \leq s_1 + \delta$, $|a - a_0| \leq \epsilon$, $|b - b_0| \leq \epsilon$, if ϵ and δ are sufficiently small.

The corner conditions (2) determine functions $s(a, b)$, $p(a, b)$ of class C^1 near the point 1, such that for each point of the corner surface

$$S_1: x = \varphi[s(a, b), a, b] \equiv \tilde{x}_1(a, b),$$

in a neighborhood of 1, the extremal of the family (3) through this point has a unique continuation direction $(p(a, b))$ in a neighborhood of the direction of E_{12} at 1.¹ There exists a space extremal tangent to the arc E_{12} at 1 which is of class C^n for $s_1 - \rho \leq s \leq s_1 + \rho$, for ρ sufficiently small, as follows from well-known existence theorems for differential equations in view of assumption (C). This extremal is the continuation extremal of E_{01} at 1. We shall denote it by E_1 . After an application of the imbedding theorems to the arc E_1 , we have

THEOREM 3:3. There exists a two-parameter family of extremals

$$(4) \quad x = \psi(s, a, b),$$

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 6.

containing the arc E_1 for $a = a_0$, $b = b_0$, $s_1 \leq s \leq s_1 + \rho$, such that the direction of each member at its point of intersection with the corner surface S_1 is the same as the continuation direction $(p(a, b))$ of the member of (3) through the point. The functions ψ , ψ_s are of class C^1 near E_1 .

The next condition has to do with the behavior of the corner surface S_1 in a neighborhood of the point 1.

VI₂. If the curve E_{03} minimizes the integral I, then the directions of E_{01} and E_{12} are not on opposite sides of the corner surface S_1 at the point 1.

This follows immediately by applying the proof of the corresponding theorem in the preceding chapter¹ to the one-parameter family of extremals of the family (3) which intersects E_{12} in a certain neighborhood of the point 1. In this proof it is assumed that the arc E_{01} is positively strong preceding the point 1.

Condition VI₂ does not exclude the case where E_{01} and E_{12} , (or E_1), are tangent to the corner surface S_1 . However, E_{01} cannot be tangent to S_1 since E_{01} contains no point conjugate to the point 0. Let us define

$$D_\phi(s, a_0, b_0) \equiv \begin{vmatrix} \phi_s & \phi_a & \phi_b \end{vmatrix}_{\substack{a=a_0 \\ b=b_0}}.$$

By condition VI₁ this determinant is not zero on $s_0 < s \leq s_1$.

Then, if the determinant

$$D_\psi(s_1, a_0, b_0) \equiv \begin{vmatrix} \psi_s & \psi_a & \psi_b \end{vmatrix}_{\substack{s=s_1 \\ a=a_0 \\ b=b_0}}$$

¹See condition VI of Part II.

is not zero it has the same sign as $D_\phi(s_1, a_0, b_0)$. In this case neither E_{01} nor E_{12} is tangent to the corner surface S_1 at the point¹ 1, and S_1 cuts the surface S in a curve C_1 through 1 which is non-singular at 1.

THEOREM 3:4. If $D_\psi(s_1, a_0, b_0) \neq 0$, there exists a one-parameter sub-family of (4), each member of which is tangent to S and contains the extremal E_1 . The points of tangency determine a curve C on S through the point 1 which is non-singular at 1.

The proof of this theorem is similar to that given by Bliss and Underhill² for the corresponding theorem for the family of extremals through the point 0 in the case where the arc E_{01} is tangent to the surface S . For convenience, we shall suppose that the sub-family of the theorem is obtained from the family (4) by setting the parameter b equal to a function $b(a)$. In view of this theorem we shall assume from this point on that $D_\psi(s_1, a_0, b_0) \neq 0$.

There exists a family of surface extremals

$$\alpha = A(\sigma, a), \quad \beta = B(\sigma, a),$$

containing the arc E_{12} for $a = a_0$, $\sigma_1 \leq \sigma \leq \sigma_2$, whose equations in space are

$$(6) \quad x = \zeta[A(\sigma, a), B(\sigma, a)] \equiv \zeta(\sigma, a),$$

where σ is the length of arc along the surface extremals so that $\sigma - \sigma(a)$ is the length of arc from the curve C .³ The functions

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 27.

²Loc. cit., p. 296.

³Ibid., p. 298.

ζ, ζ_σ are of class C^1 near E_{12} and along the curve C we have the identities

$$\psi[\sigma(a), a, b(a)] = \zeta[\sigma(a), a], \quad \psi_s[\sigma(s), a, b(a)] = \zeta_\sigma[\sigma(a), a].$$

That is, near the point 1 these surface extremals are tangent to the sub-family of space extremals described in Theorem 3:4. There exists a family of space extremals tangent to the surface extremals (6) with equations of the form

$$(7) \quad x = \bar{\Phi}(s, \sigma, a),$$

where $s - \sigma$ is the length of arc from the point of tangency, i.e., $\bar{\Phi}(\sigma, \sigma, a) = \zeta(\sigma, a)$ and $\bar{\Phi}_s(\sigma, \sigma, a) = \zeta_\sigma(\sigma, a)$ for $|a - a_0| \leq \epsilon$, $s_1 - \delta \leq \sigma \leq s_2 + \delta$, if ϵ and δ are sufficiently small.¹ The functions $\bar{\Phi}$, $\bar{\Phi}_s$ are of class C^1 near $s = \sigma$, $s_1 \leq \sigma \leq s_2$, $a = a_0$.

Since the corner equations (2) are satisfied at the point 2 on E_{03} , they determine functions $s(\sigma, a)$, $p(\sigma, a)$ of class C^1 near (s_2, σ_2, a_0) , such that for each point of the surface

$$S_2: x = \bar{\Phi}[s(\sigma, a), \sigma, a] \equiv \tilde{x}_2(\sigma, a),$$

and the corresponding extremal of the family (7) there exists a unique continuation direction $[p(\sigma, a)]$ in a neighborhood of the direction of E_{23} at the point 2.² For the parameter values $\sigma = \sigma_2$, $a = a_0$ the direction $[p(\sigma, a)]$ coincides with that of the arc E_{23} at 2. The surface S_2 lies entirely on the same side of the surface S in a neighborhood of the point 2 as the family (7) defining it.

From an application of the embedding theorems, we have the existence of a family of space extremals

¹Ibid., p. 299.

²Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 6.

$$(8) \quad x = \bar{\Psi}(s, \sigma, a),$$

containing the extremal E_{23} for $s_2 \leq s \leq s_3$, $\sigma = \sigma_2$, $a = a_0$. The functions $\bar{\Psi}$, $\bar{\Psi}_s$, $\bar{\Psi}_{ss}$ are of class C^1 near E_{23} and the direction $[p(\sigma, a)]$ coincide with the direction $[\bar{\Psi}_s(s(\sigma, a), \sigma, a)]$ for (σ, a) sufficiently near (σ_2, a_0) .

Let

$$(9) \quad x = \psi[s, a, b(a)] \equiv \psi_0(s, a)$$

be the equations of the one-parameter sub-family of space extremals from the family (4) which are tangent to the surface S at points of the curve C . Then the relation between the curves C and C_1 on S is determined by the following theorem:

THEOREM 3:5. If the points of the curve C do not precede the corresponding points of the corner surface along extremals of the family (9), then C determines a one-parameter family of extremals through the point O tangent to S at points of C . This will be the case, for example, when $\mathcal{G}(\zeta, \zeta', p) \geq 0$ along the surface extremals in a neighborhood of the point l and for all $(p) \neq (0)$.

The first part of the theorem is evidently true, since in that case the family (9) connects up with a part of the family (3) at points of the corner surface S_1 before touching the surface S .

In order to prove the last part of the theorem, we suppose that it is not true. Then choose a member of the family (9) sufficiently near the arc E_1 which is tangent to S , in this case, preceding its point of intersection with the corner surface S_1 . Since, from Theorems 1:1 and 3:2, the function $\mathcal{G}[\psi_0(s, a), \psi'_0(s, a), p]$ is an increasing function of its argument s in a neighborhood N of the point l and is equal to zero at S_1 for the

directions of the extremaloid there, we see that $\mathcal{G}[\psi_0(s, a), \psi'_0(s, a), p]$ has to be negative preceding S_1 for some direction (p) . Thus if we choose our extremal near enough to E_1 so that its point of tangency with the surface S lies in the neighborhood N , then we see that $\mathcal{G}[\psi_0(s, a), \psi'_0(s, a), p]$ will be negative for some direction (p) at this point of tangency with the surface extremal. This contradicts the hypothesis that $\mathcal{G} \geq 0$ along the surface extremals in a neighborhood of the point 1.

THEOREM 3:6. If at a point of a surface extremal we have $\mathcal{G}(\zeta, \zeta', p) > 0$ except for (p) in the directions (ζ') and $(\bar{\kappa}')$, and $\Omega_0(\zeta, \zeta', \bar{\kappa}') > 0$ ($\Omega_0 < 0$), then the space extremal tangent to the surface extremal commences (ceases) to be positively strong at that point.

The proof of this theorem is similar to that of the analogous theorem for the plane case.¹ It follows then that the space extremal E_1 commences to be positively strong at the point 1 and the space extremal tangent to E_{12} at the point 2, which we shall denote by E_2 , ceases to be positively strong at 2. The surface extremals really have nothing to do with the theorem, but we shall have occasion to use it only as it is stated.

THEOREM 3:7. If $\mathcal{G}(\zeta_2, \zeta'_2, p) > 0$ except for (p) in the directions (ζ'_2) and $(\bar{\kappa}_2')$, and $\Omega_0(\zeta_2, \zeta'_2, \bar{\kappa}_2') < 0$, a necessary and sufficient condition that the points of the corner surface S_2 do not precede the corresponding points of the surface S along the extremals of the family (7) tangent to S in some neighborhood of the point 2, is that $\mathcal{G}(\zeta, \zeta', p) \geq 0$ along the surface extremals in some neighborhood of the point 2 for all $(p) \neq (0)$.

¹See the proof of Lemma 2:1.

From the relation in Theorem 1:1, we see that there exists a neighborhood $N_\epsilon(s_2, \sigma_2, a_0, \bar{x}_2')$ such that $\mathfrak{G}(\Phi(s, \sigma, a), \Phi_s(s, \sigma, a), p) \equiv \mathfrak{G}(s, \sigma, a, p)$ is a decreasing function of s in (s, σ, a, p) in N_ϵ . Suppose that the condition of the theorem is not sufficient. Choose an extremal of the family tangent to the surface S so near E_2 that the element $[s'(\sigma', a'), \sigma', a', \bar{x}']$ is in N_ϵ , where $s'(\sigma', a')$ is the value of s at the corner surface S_2 on this extremal. But we know that $\mathfrak{G}[s'(\sigma', a'), \sigma', a', \bar{x}'] = 0$ on account of the corner conditions. Therefore, $\mathfrak{G}(s, \sigma', a', \bar{x}') < 0$ for s in N_ϵ and greater than $s'(\sigma', a')$. Thus $\mathfrak{G}(\sigma', \sigma', a', \bar{x}') < 0$, which contradicts the hypothesis that $\mathfrak{G} \geq 0$ along the surface extremals in some neighborhood of the point 2.

In order to prove that the condition is necessary suppose that the points of the corner surface S_2 do not precede the corresponding points of the surface S along the extremals tangent to S . We wish to show that $\mathfrak{G}(\zeta, \zeta', p) \geq 0$ in a neighborhood of the point 2 for all $(p) \neq (0)$. The extremal E_2 is positively strong preceding the corner surface S_2 by Theorem 3:6. Then, by Theorem 1:2, each member of the family of space extremals containing E_2 is positively strong preceding S_2 for (σ, a) sufficiently near (σ_2, a_0) and ceases to be strong at the corner. But we have supposed that the points of S_2 do not precede the corresponding points of S along the extremals. Thus, at least $\mathfrak{G}(\zeta, \zeta', p) \geq 0$ for (σ, a) sufficiently near (σ_2, a_0) .

3:4. Analogue of the Jacobi condition. The conditions VI_1 and VI_2 of the two preceding sections played an important part in the construction of the desired families of extremals in a neighborhood of the point 1. In other words it seems necessary to make use of the Jacobi condition for the arc E_{01} in proving

the same condition for the arcs E_{12} and E_{23} . In deriving the Jacobi condition for the arcs E_{12} and E_{23} , we will find it necessary to make some rather restrictive assumptions in addition to the usual ones in order to carry through the usual proofs.

VI₃. If the points of the curve C do not precede the corresponding points of S_1 along the extremals (4); if E_{03} contains a point of contact with an enveloping curve d of the family (6) of extremals on S either between the points 1 and 2 or at 2 and d has a regressive branch at this point; then the curve E_{03} cannot minimize the integral I.

The equation which defines the envelope d on the surface S is

$$\Delta(\sigma, a) \equiv \begin{vmatrix} A_\sigma & A_a \\ B_\sigma & B_a \end{vmatrix} = 0,$$

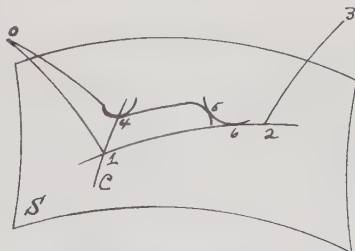
where A and B are the functions defining the extremals on the surface. If this equation is satisfied at a point $(\sigma, a) = (\sigma_6, a_0)$ of the arc E_{12} , the derivative Δ_σ will be different from zero at that point.¹ The above equation has therefore a solution

$$\sigma = \Sigma(u), \quad a = T(u),$$

reducing to (σ_6, a_0) for $u = a_0$ and of class C' near that value. The equations in space of the envelope have the form

$$d: x = \zeta[\Sigma(u), T(u)].$$

The parameter u is chosen as one of the values $\pm a$ so as to make the positive direction along d the same as that of the



¹Fliss and Underhill, loc. cit., p. 302.

surface extremal tangent to it. For an arbitrary value of u near u_0 , the corresponding value of a defines an extremaloid E_{05} which is tangent to d at the point 5 as shown in the figure, and d_{56} touches E_{03} at the point 6. Then from Theorem 1:4, we see that the function

$$I(u) = I(E_{05} + d_{56} + E_{63})$$

is a constant for u sufficiently near u_0 . An argument like that used in the plane case shows that d_{56} cannot be an arc of an extremal on the surface S . Consequently, since $I(a_0) = I(E_{03})$, it follows that the curve E_{03} cannot minimize the integral I .

VI₄. If the points of the curve C do not precede the corresponding points of the corner surface S_1 along the extremals (4); if $\mathcal{E}(\zeta, \zeta', p) \geq 0$ in a neighborhood of the point 2 and equals zero at the point 2 only for (p) in the directions (ζ') and $(\bar{x}_2')$; if the curve E_{03} contains a point of contact between the points 2 and 3 or at 3 with a non-singular enveloping surface D of the family (8) of extremals containing the arc E_{23} ; then the curve E_{03} cannot minimize the integral I .

The equations which define the enveloping surface D of the family of extremals (8) is

$$\Delta(s, \sigma, a) \equiv |\psi_s \ \psi_\sigma \ \psi_a| = 0.$$

If this equation is satisfied at a point 8 of the arc E_{23} for which $(s, \sigma, a) = (s_8, \sigma_8, a_8)$ and defines a non-singular enveloping surface D , then the argument of Mason and Bliss¹ shows that there is a curve d on D with equations

$$d: x = x(u)$$

¹Loc. cit., p. 450.

passing through the point 8 for $u = u_0$, which is tangent to a one-parameter family of extremals containing E_{23} at points defined by equations of the form

$$s = S(u), \quad \sigma = \Sigma(u), \quad a = A(u),$$

where S, Σ, A reduce to s_8, σ_8, a_8 for $u = u_0$. The parameter u is chosen so that the positive direction on the curve d coincides with the positive direction of the extremal tangent to d at the point defined by these equations. An arbitrary value of u near u_0 defines an extremaloid E_{04} of the set (9) tangent to a surface extremal E_{45} of the family (6) at a point 4 of the curve C ; E_{45} is tangent at 5 to the extremal E_{56} defined by the equations

$$(10) \quad x = \Phi(s, \Sigma, A);$$

E_{56} meets E_{67} , defined by equations of the form

$$(11) \quad x = \Psi(s, \Sigma, A),$$

at the corner surface S_2 ; and finally this last curve is tangent to d_{78} at the point 7. When u is set equal to v , the families (9), (6), (10), (11) and d_{78} are of the kind described in Theorem 1:4. Thus the function

$$I(u) = I(E_{04} + E_{45} + E_{56} + E_{67} + d_{78} + E_{83})$$

is a constant for u sufficiently near u_0 . Since $I(u_0) = I(E_{03})$ and the envelope d cannot be an arc of an extremal, it follows that the curve E_{03} cannot minimize the integral I .

3:5. Modification of the original hypotheses. At the beginning of this chapter we assumed, for convenience, that the arcs E_{01} and E_{23} were of class C^1 . We may assume that these arcs are composed of a finite number of arcs each of class C^1 .

Since these curves are by hypothesis interior to the admissible region, we may assume the results of Graves¹ with the restrictions imposed by the boundary. Therefore, we have the existence of a family of extremaloids through the point O ,

$$(12) \quad x = \chi(s, a, b),$$

containing the extremaloid $E_{01} + E_1$ for $a = a_0$, $b = b_0$, $s_0 \leq s \leq s_1 + \rho$. This family has the same continuity properties as the family (3) except at the corner surfaces.² Then

$$(13) \quad x = \chi(s, a, b(a)) \equiv \chi_*(s, a)$$

is the one-parameter sub-family of extremaloids which has each member tangent to the surface S at points of the curve C . Also there exists a family of extremaloids

$$(14) \quad x = \psi(s, \sigma, a),$$

containing the extremaloid E_{23} for $a = a_0$, $\sigma = \sigma_2$, $s_2 \leq s \leq s_3$, and having the same continuity properties as the family (8) except at points of the corner surfaces.

If we combine the results of Graves with the results of this chapter, we have the following statement of the conditions of the preceding sections appropriately modified to be of use in proving sufficiency theorems:

I. E_{01} and E_{23} are extremaloids without multiple points and such that $\Omega_0 \neq 0$ at the corners of E_{03} ;

II. $I^{(x_1)}_{X_1} = \lambda(s)$, ($i = 1, 2, 3$), along E_{12} ;

¹Discontinuous solutions, American Journal of Mathematics, LII (1930), pp. 1-25.

²Graves, ibid., p. 6.

III'. $\mathcal{G}(s) > 0$ along E_{12} ;

IV'. E_{03} is positively strong and uniquely determined by each of its elements (x, x') ; $\mathcal{G}(\zeta, \zeta', p) \geq 0$ along the surface extremals of the family (6) in certain neighborhoods of the points 1 and 2;

V. $\mathcal{G}(\zeta, \zeta', \bar{x}') = 0$ at the points 1 and 2 on E_{03} ;

VI'. The determinant

$$\Delta_1(s, a_0, b_0) = \left| \chi_s \chi_a \chi_b \right|_{\substack{s=a_0 \\ b=b_0}}$$

does not vanish nor change sign on $s_0 < s \leq s_1 + \rho$; the determinant

$$\Delta_2(\sigma, a_0) = \left| \begin{array}{cc} A_\sigma & A_a \\ B_\sigma & B_a \end{array} \right|_{a=a_0}$$

does not vanish on $\sigma_1 \leq \sigma \leq \sigma_2$; and the determinant

$$\Delta_3(s, \sigma_2, a_0) = \left| \begin{array}{ccc} \Psi_s & \Psi_\sigma & \Psi_a \end{array} \right|_{\substack{\sigma=\sigma_2 \\ a=a_0}}$$

does not vanish nor change sign on $s_2 \leq s \leq s_3$.

3:6. Existence of a field of extremaloids. We shall assume here that the conditions described in the preceding section hold together with the following:

VII. $F_1(x, p) \neq 0$ at the point 0 for all directions (p) ;

VIII. The extremal arc of E_{01} preceding the corner at the point 1 is not tangent to E_{12} at 1.

Condition VIII excludes the case which is analogous to the case considered in Section 2:10 for the problem in the plane.

THEOREM 3:8. There exists a field of extremaloids $\mathcal{F}_1$ about the extremaloid $E_{01} + E_1$, such that through each point of $\mathcal{F}_1$, other than the point 0, there passes a unique extremaloid of the family (12) defined by the single-valued functions

$$s = s(x), \quad a = a(x), \quad b = b(x)$$

which are of class C' in $\mathcal{F}_1$ except at the point 0 and at corners.¹

The region

$$\sigma \leq s \leq \sigma + \rho, \quad \sigma(a) \leq \sigma \leq s_2 + h, \quad (|a - a_0| \leq \epsilon_2),$$

defines by equations (7) a field $\mathcal{F}_2$ of extremals tangent to the surface extremals (6) on the surface S , provided ρ and ϵ are sufficiently small. The single-valued functions

$$s = s(x), \quad \sigma = \sigma(x), \quad a = a(x)$$

defined by the equations (7) are of class C' in $\mathcal{F}_2$ and continuous on the boundary.²

There exists a field of extremaloids $\mathcal{F}_3$ about the extremaloid E_{23} , such that through each point of $\mathcal{F}_3$ there passes a unique extremaloid of the family (8) defined by the single-valued functions

$$s = s(x), \quad \sigma = \sigma(x), \quad a = a(x),$$

which are of class C' in $\mathcal{F}_3$ except at corners.³

Now, let $\mathcal{F}$ denote the x -points determined by equations (12) preceding and including the point of intersection of each member of the family with the surface S for $s_0 + \delta \leq s \leq s_1(a, b) + \rho$, $|a - a_0| \leq \epsilon_1$, $|b - b_0| \leq \epsilon_1$; the x -points in the δ neighborhood of the point 0, where δ is taken sufficiently small; the x -points determined by equations (7) for $\sigma \leq s \leq \sigma + \rho$,

¹Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 25.

²Bliss and Underhill, loc. cit., p. 304.

³Graves, Discontinuous solutions, American Journal of Mathematics, LII (1930), p. 25.

$\sigma(a) < \sigma \leq \sigma_2 - h$, $|a - a_0| \leq \epsilon_2$, and also for $\sigma \leq s \leq s_2(\sigma, a)$, $\sigma_2 - h < \sigma \leq \sigma_2 + h$, $|a - a_0| \leq \epsilon_2$, where $\sigma(a)$ is the value of σ at the curve C and $s_2(\sigma, a)$ is the value of s at the corner surface S_2 ; the x -points determined by equations (14) for $s_2(\sigma, a) < s \leq s_3 + \epsilon_3$, $|\sigma - \sigma_2| \leq \epsilon_4$, $|a - a_0| \leq \epsilon_5$. Then choose the constants ϵ_1 , ϵ_3 , and h sufficiently small independently. Choose ϵ_2 and ϵ_4 sufficiently small so that $\epsilon_2 \leq \epsilon_1$ and $\epsilon_4 \leq h$. Finally, choose $\epsilon_5 \leq \epsilon_2$ and ρ sufficiently small so that the regions of $\mathcal{F}$ connect up and do not overlap.

THEOREM 3:9. If the region $\mathcal{F}$ is sufficiently restricted, then it is a field of extremaloids about E_{03} adjoined to S . Through each point of $\mathcal{F}$ there passes a unique curve E which also passes through the point O and lies on the same side of S as E_{01} and E_{23} . This curve E is composed of a space extremaloid not tangent to S , or of a space extremaloid tangent to S plus an arc of a surface extremal plus a space extremaloid tangent to the surface extremal.

This follows directly from Theorem 3:8 and the fact that condition IV' is sufficient in order that the families of extremals in certain neighborhoods of the points 1 and 2 can be pieced together to form admissible extremaloids. The curve E of the theorem does not go outside the region $\mathcal{F}$ due to the manner in which the constants determining the region were chosen.

THEOREM 3:10. Each curve E of the field $\mathcal{F}$ described in Theorem 3:9 is positively strong if $\mathcal{F}$ is sufficiently restricted, except possibly at points of the surface extremals in neighborhoods of the points 1 and 2 at which the function $\mathcal{G}_1$ may vanish.

From Theorem 1:2 and condition IV' it follows that each member of the families (12) and (14) in $\mathcal{F}$ is positively strong if $\mathcal{F}$ is sufficiently restricted, since E_1 has been shown to be

positively strong beyond the point 1. From condition IV' the arc E_{12} is positively strong for all directions in space except at the points 1 and 2. From the continuity of the function $\mathfrak{G}_1$, we see that each member of the family of surface extremals containing E_{12} is positively strong for all directions in space for $|a - a_0| \leq \epsilon_2$ for ϵ_2 sufficiently small, except possibly at points in certain neighborhoods $N_{1\epsilon}(\sigma_1, a_0)$ and $N_{2\epsilon}(\sigma_2, a_0)$ of the points 1 and 2 respectively. If $\mathfrak{G}_1 = 0$ along this family at a point of $N_{1\epsilon}(\sigma_1, a_0)$ for some direction, we have by Theorem 3:6 that the space extremal tangent to S at that point commences to be positively strong. If $\mathfrak{G}_1 = 0$ at a point of $N_{2\epsilon}(\sigma_2, a_0)$, this point is also a point of the corner surface S_2 , since $(\bar{x}_2')$ is assumed to be the unique continuation direction of the space extremal tangent to E_{12} at the point 2, which we have denoted by E_2 . For, by Lemma 1:2 each extremal of the family tangent to S in a neighborhood of 2 has a unique continuation direction at S_2 . Also each member of this family sufficiently near E_2 is positively strong preceding S_2 . But by Theorem 3:7 the points of S_2 follow the corresponding points of S along the space extremals tangent to S . Therefore, each member of the family of surface extremals is positively strong for all directions in space at points of $N_{2\epsilon}(\sigma_2, a_0)$ except at points where S_2 and S coincide. S and S_2 may have any number of points and regions in common in the neighborhood $N_{2\epsilon}(\sigma_2, a_0)$. However, S_2 has no point of E_{12} preceding 2 in common with S , since we have assumed E_{12} to be positively strong preceding 2. Finally, each space extremal tangent to a positively strong surface extremal is itself positively strong near their point of contact. Thus each curve E of the field $\mathfrak{F}$ is positively strong except possibly for arcs of surface extremals in the neighborhoods $N_{1\epsilon}(\sigma_1, a_0)$ and $N_{2\epsilon}(\sigma_2, a_0)$ for which the

function $\mathfrak{G}_1$ is at least non-negative.

3:7. Sufficient conditions for a minimum. We shall prove a general sufficiency theorem here which is analogous to the sufficiency Theorem 2:12 for the plane case. Then it will follow from this theorem that the conditions stated in Sections 3:5 and 3:6 are sufficient in order that the curve E_{03} minimize the integral I.

THEOREM 3:11. Suppose that the curve E_{03} is "imbedded" in a field $\mathfrak{F}$ of extremaloids adjoined to S such that for each point (x) in $\mathfrak{F}$ there exists a unique curve E satisfying the following conditions:

- (1) E joins the point (x) to the point O;
- (2) E is of class C^1 at least except for points at corners;
- (3) E is composed of a space extremaloid not tangent to S or of a space extremaloid tangent to a surface extremal plus an arc of the surface extremal plus an arc of a space extremaloid tangent to the surface extremal;

(4) each space extremaloid arc of E is positively strong.
Then $I(V_{03}) > I(E_{03})$ for every admissible curve V in $\mathfrak{F}$ distinct from the curve E_{03} .

Consider the extremaloid integral

$$W(x) = \int_0^{t(x)} F(x, x') dt.$$

Except at the point O, this function W has the partial derivatives

$$W_{x_1}(x) = F_{x_1}(x, p),$$

where (p) denotes the direction components of the extremal of the field through (x). The proof that the function W(x) has these partial derivatives at corner surfaces, at points defined by the sub-family

$$x = \psi[s, a, b(a)] \equiv \psi_0(s, a)$$

for $\sigma(a) \leq s \leq \sigma(a) + \rho$, $\sigma = \sigma(a)$, $|a - a_0| \leq \epsilon$, and at points of the surface S is similar to that in the proof of Theorem 2:12 for the analogous function in the plane case. Let V_{03} represent any admissible comparison curve in $\mathcal{F}$ with equations of the form

$$V_{03}: x = X(\nu) \quad (\nu_0 \leq \nu \leq \nu_3).$$

Then, the function

$$W(\nu) = W[X(\nu)] + \int_{\nu}^{\nu_3} F(X, X') d\nu$$

has a derivative

$$W'(\nu) = - \mathfrak{E}(x, p, X')$$

except at the point 0 and the corners of the curve V_{03} . This says that the function $W(\nu)$ is decreasing on the interval (ν_0, ν_3) since the function $\mathfrak{E}(x, p, X')$ is non-negative. Thus,

$$\Delta I \equiv I(V_{03}) - I(E_{03}) = W(\nu_0) - W(\nu_3) > 0$$

unless the derivative $W'(\nu)$ is identically zero on the interval (ν_0, ν_3) . In this case the curves E_{03} and V_{03} coincide throughout.¹

THEOREM 3:12. If the conditions I, II, III', IV', V, VI', VII and VIII are satisfied along the curve E_{03} , then $I(V_{03}) > I(E_{03})$ for every admissible curve V in a sufficiently small neighborhood of the curve E_{03} distinct from E_{03} .

This follows immediately from the general sufficiency Theorem 3:11, since it was shown in Section 3:6 that these conditions insured the existence of a field $\mathcal{F}$ having the properties required in the sufficiency theorem.

¹Bliss and Underhill, loc. cit., p. 307.

